EGM 3401

Theory Assignment #2

Spring 2015

Due Date: 17 March 2015 (In Class)

Question 1

Consider the simple pendulum problem in Chapter 3 on pages 157–163 of the textbook. Using the definition of angular momentum, ${}^{\mathcal{N}}\mathbf{H}_Q$, relative to an arbitrary point Q, show that the component of the reaction force exerted by the hinge in the direction transverse to the arm must be zero.

Hint: in order to solve this problem, take the system as being the arm and the particle, then choose appropriate reference point about which to compute the angular momentum.

Question 2

Let $\mathcal{A}$ be an inertial reference frame, and let $\mathcal{B}$ be be a reference frame that translates with constant velocity relative to reference frame $\mathcal{A}$. Prove that $\mathcal{B}$ is also inertial reference frame.

Question 3

Let ${}^{\mathcal{A}}\boldsymbol{\omega}^{\mathcal{B}}$ be the angular velocity of reference frame $\mathcal{B}$ relative to reference frame $\mathcal{A}$. Show that the operation of taking the vector product of ${}^{\mathcal{A}}\boldsymbol{\omega}^{\mathcal{B}}$ with an arbitrary vector $\mathbf{b}$ is a tensor and show the matrix representation of this tensor in a basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$, where $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ is fixed in $\mathcal{B}$.

Question 4

Let T be the kinetic energy of a particle of mass m and let $\mathbf{r}$ be the position of the particle measured relative to an inertially fixed point. Suppose now that the position of the particle is parameterized in terms of three independent scalar quantities (q_1, q_2, q_3) and time, that is, $\mathbf{r} = \mathbf{r}(q_1, q_2, q_3, t)$. Finally, let $\mathbf{v} = \dot{\mathbf{r}} = d\mathbf{r}/dt$ be the inertial velocity and let $\mathbf{a} = \dot{\mathbf{v}} = d\mathbf{v}/dt = \ddot{\mathbf{r}} = d^2\mathbf{r}/dt^2$ be the inertial acceleration of the particle (that is, assume for this problem that the notation d/dt refers to a rate of change taken in the inertial reference frame). Prove that

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_j} - \frac{\partial T}{\partial q_j} = \mathbf{F} \cdot \frac{\partial \mathbf{r}}{\partial q_j}, \quad (j = 1, 2, 3),$$

where $\mathbf{F}$ is the resultant force acting on the particle.
Hint: In your solution, use the fact that the inertial velocity can be written as

$$\mathbf{v} = \dot{\mathbf{r}} = \frac{d\mathbf{r}}{dt} = \sum_{i=1}^{3} \frac{\partial \mathbf{r}}{\partial q_i}\dot{q}_i + \frac{\partial \mathbf{r}}{\partial t}$$

Question 5

The moment of momentum of a particle relative to an arbitrary point Q in an inertial reference frame $\mathcal{N}$ is defined as

$${}^{\mathcal{N}}\mathbf{L}_Q = (\mathbf{r} - \mathbf{r}_Q) \times m\,{}^{\mathcal{N}}\mathbf{v},$$

where $\mathbf{r}$ is the position of the particle measured relative to a point fixed in $\mathcal{N}$ and ${}^{\mathcal{N}}\mathbf{v}$ is the velocity of the particle as viewed by an observer fixed in $\mathcal{N}$. Show that relative to Q as

$$\frac{{}^{\mathcal{N}}d}{dt}\left({}^{\mathcal{N}}\mathbf{L}_Q\right) = (\mathbf{r} - \mathbf{r}_Q) \times \mathbf{F} - {}^{\mathcal{N}}\mathbf{v}_Q \times m\,{}^{\mathcal{N}}\mathbf{v},$$

where $\mathbf{F}$ is the resultant force acting on the particle.

Question 6

Consider a system of n particles of mass $(m_1, \ldots, m_n)$. Let $(\mathbf{r}_i, {}^{\mathcal{N}}\mathbf{v}_i, {}^{\mathcal{N}}\mathbf{a}_i)$ be the position, inertial velocity, and inertial acceleration of particle $i \in [1, \ldots, n]$. Prove the following statements:

(a) $\sum_{i=1}^{n} m_i \mathbf{r}_i - m\bar{\mathbf{r}} = \mathbf{0}$,

(b) $\sum_{i=1}^{n} m_i {}^{\mathcal{N}}\mathbf{v}_i - m {}^{\mathcal{N}}\bar{\mathbf{v}} = \mathbf{0}$,

(c) $\sum_{i=1}^{n} m_i {}^{\mathcal{N}}\mathbf{a}_i - m {}^{\mathcal{N}}\bar{\mathbf{a}} = \mathbf{0}$,

where $m = \sum_{i=1}^{n} m_i$, and $(\bar{\mathbf{r}}, {}^{\mathcal{N}}\bar{\mathbf{v}}, {}^{\mathcal{N}}\bar{\mathbf{a}})$, are, respectively, the position, velocity, and acceleration of the center of mass of the system.

Question 7

Consider a system of n particles of mass $(m_1, \ldots, m_n)$. Furthermore, let ${}^{\mathcal{N}}\mathbf{a}_i$, $(i = 1, \ldots, n)$ be the acceleration of particle $i \in [1, \ldots, n]$ as viewed by an observer in the inertial reference frame $\mathcal{N}$. Finally, let $\mathbf{R}_i = \mathbf{F}_i + \sum_{j=1}^{n} \mathbf{f}_{ij}$ be the resultant force acting on particle $i \in [1, \ldots, n]$, where $\mathbf{F}_i$ is the resultant *external* force acting on particle i and $\mathbf{f}_{ij}$ is the force exerted by particle j on particle i. Show that

$$\mathbf{F} = \sum_{i=1}^{n} \mathbf{F}_i = m {}^{\mathcal{N}}\bar{\mathbf{a}}, \tag{1}$$

where ${}^{\mathcal{N}}\bar{\mathbf{a}}$ is the acceleration of the center of mass of the system as viewed by an observer in the inertial reference frame $\mathcal{N}$.

Question 8

Consider a system of n particles of mass $(m_1, \ldots, m_n)$. Let $(\mathbf{r}_i, {}^{\mathcal{N}}\mathbf{v}_i, {}^{\mathcal{N}}\mathbf{a}_i)$ be the position, inertial velocity, and inertial acceleration of particle $i \in [1, \ldots, n]$. Furthermore, let ${}^{\mathcal{N}}\mathbf{H}_Q$ be the angular momentum of the system relative to an arbitrary point Q, where ${}^{\mathcal{N}}\mathbf{H}_Q$ is defined as

$${}^{\mathcal{N}}\mathbf{H}_Q = \sum_{i=1}^{n} (\mathbf{r}_i - \mathbf{r}_Q) \times m_i ({}^{\mathcal{N}}\mathbf{v}_i - {}^{\mathcal{N}}\mathbf{v}_Q).$$

Show that

$${}^{\mathcal{N}}\mathbf{H}_Q = {}^{\mathcal{N}}\bar{\mathbf{H}} + (\mathbf{r}_Q - \bar{\mathbf{r}}) \times m ({}^{\mathcal{N}}\mathbf{v}_Q - {}^{\mathcal{N}}\bar{\mathbf{v}}),$$

where $m = \sum_{i=1}^{n} m_i$ is the total mass of the system and ${}^{\mathcal{N}}\bar{\mathbf{H}}$ is the angular momentum relative to the center of mass of the system.

Question 9

Consider a system of n particles of mass $(m_1, \ldots, m_n)$. Furthermore, let ${}^{\mathcal{N}}\mathbf{v}_i$, $(i = 1, \ldots, n)$ be the velocity of particle $i \in [1, \ldots, n]$ as viewed by an observer in the inertial reference frame $\mathcal{N}$. The angular momentum of a system of n particles relative to an arbitrary point is defined as

$${}^{\mathcal{N}}\mathbf{H}_Q = \sum_{i=1}^{n} (\mathbf{r} - \mathbf{r}_Q) \times m_i ({}^{\mathcal{N}}\mathbf{v}_i - {}^{\mathcal{N}}\mathbf{v}_Q).$$

Show the following results:

(i)

$$\frac{{}^{\mathcal{N}}d}{dt}\left({}^{\mathcal{N}}\mathbf{H}_Q\right) = \mathbf{M}_Q - (\bar{\mathbf{r}} - \mathbf{r}_Q) \times m^{\mathcal{N}}\mathbf{a}_Q,$$

(ii)

$$\frac{{}^{\mathcal{N}}d}{dt}\left({}^{\mathcal{N}}\bar{\mathbf{H}}\right) = \bar{\mathbf{M}},$$

(iii)

$$\frac{{}^{\mathcal{N}}d}{dt}\left({}^{\mathcal{N}}\mathbf{H}_O\right) = \mathbf{M}_O,$$

where $\mathbf{M}_Q$, $\bar{\mathbf{M}}$, and $\mathbf{M}_O$ are, respectively, the moments due to all *external* forces relative to the artbitrary point Q, the center of mass of the system, and the inertially fixed point O, and ${}^{\mathcal{N}}\mathbf{a}_Q$ is the acceleration of point Q as viewed by an observer in the inertial reference frame $\mathcal{N}$.

Question 10

Consider a system of n particles of mass $(m_1, \ldots, m_n)$. Furthermore, let ${}^{\mathcal{N}}\mathbf{v}_i$, $(i = 1, \ldots, n)$ be the velocity of particle $i \in [1, \ldots, n]$ as viewed by an observer in the inertial reference frame $\mathcal{N}$. Show that the kinetic energy of the system can be written as

$$T = \frac{1}{2}m^{\mathcal{N}}\bar{\mathbf{v}} \cdot {}^{\mathcal{N}}\bar{\mathbf{v}} + \frac{1}{2}\sum_{i=1}^{n} m_i({}^{\mathcal{N}}\mathbf{v}_i - {}^{\mathcal{N}}\bar{\mathbf{v}}) \cdot ({}^{\mathcal{N}}\mathbf{v}_i - {}^{\mathcal{N}}\bar{\mathbf{v}}),$$

where ${}^{\mathcal{N}}\bar{\mathbf{v}}$ is the inertial velocity of the center of mass of the system.