Question 1

Let $\mathbf{A} \in \mathbb{R}^{n \times n}$ be a real symmetric $n \times n$ matrix. Prove the following:

- The eigenvalues of $\mathbf{A}$ are real
- The eigenvectors corresponding to distinct eigenvalues are orthogonal

Assume now that in the case of repeated eigenvalues the eigenvectors are *still* orthogonal (i.e., for $\lambda_i = \lambda_j$ it is still the case that the corresponding eigenvectors $\mathbf{v}_i$ and $\mathbf{v}_j$ are orthogonal). Prove the following properties of the eigenvector matrix $\mathbf{V} = [\mathbf{v}_1 \cdots \mathbf{v}_n]$:

- $\mathbf{V}$ is an *orthoogonal* matrix, i.e., $\mathbf{V}^{-1} = \mathbf{V}^T$
- $\mathbf{V}^T\mathbf{A}\mathbf{V} = \mathbf{\Lambda}$ where $\mathbf{\Lambda}$ is a diagonal matrix of eigenvalues

Question 2

Let $\mathbf{A} \in \mathbb{R}^{m \times n}$ be a real matrix. Prove the following results:

- The general solution of the linear system $\mathbf{A}\mathbf{x} = \mathbf{b}$ is $\mathbf{x} = \mathbf{x}_h + \mathbf{x}_p$ where $\mathbf{x}_h$ is the *homogeneous solution* and satisfies $\mathbf{A}\mathbf{x}_h = \mathbf{0}$ and $\mathbf{x}_p$ is the *particular solution* and satisfies $\mathbf{A}\mathbf{x}_p = \mathbf{b}$
- In the case where $\mathbf{A}$ is *square*, the unique solution to the linear system $\mathbf{A}\mathbf{x} = \mathbf{b}$ is $\mathbf{x} = \mathbf{A}^{-1}\mathbf{b}$ and this solution exists if and only if the matrix $\mathbf{A}$ is nonsingular.

Question 3

Let $\mathbf{A}, \mathbf{B} \in \mathbb{R}^{n \times n}$ be constant $n \times n$ matrices. Furthermore, let $a(s)$ and $b(s)$ each be arbitrary polynomials in s of degree m. Prove the following result:

$$a(\mathbf{A})b(\mathbf{B}) = b(\mathbf{B})a(\mathbf{A}) \Longleftrightarrow \mathbf{A}\mathbf{B} = \mathbf{B}\mathbf{A}$$

Question 4

Let $\mathbf{A} \in \mathbb{R}^{n \times n}$ be a constant $n \times n$ matrix. Prove that the matrix exponential $e^{\mathbf{A}}$ can be written as a linear combination of powers of $\mathbf{A}$ of degree $n-1$ or less, i.e.,

$$e^{\mathbf{A}} = \sum_{k=0}^{n-1} \alpha_k \mathbf{A}^k$$

where $\alpha_k, \ (k = 0, \ldots, n-1)$ are real numbers.

Question 5

Let $\mathbf{B} \in \mathbb{R}^{n \times n}$ be an $n \times n$ real matrix. Furthermore, let $\mathbf{L}(\mathbf{A}) = \mathbf{AB} + \mathbf{BA}$ be a matrix operator whose input argument is $\mathbf{A}$. Show that $\mathbf{L}$ is a linear operator.

Question 6

Let $\mathbf{A} \in \mathbb{R}^{n \times n}$ be an $n \times n$ real square matrix. Furthermore, let $\mathbf{x}$ be an eigenvector of $\mathbf{A}$ with eigenvalue λ. Prove that

$$\mathbf{A}^k \mathbf{x} = \lambda^k \mathbf{x}$$

for all $k \geq 1$.

Question 7

Let $p(s)$ be a polynomial of degree n in $s \in \mathbb{C}$. Furthermore, let $\mathbf{A} \in \mathbb{R}^{n \times n}$ be an $n \times n$ real square matrix. Finally, let λ be an eigenvalue of $\mathbf{A}$ with eigenvector $\mathbf{x}$. Prove that $p(\lambda)$ is an eigenvalue of $p(\mathbf{A})$ with eigenvector $\mathbf{x}$.

Question 8

Let $\mathbf{A} \in \mathbb{R}^{n \times n}$ be an $n \times n$ real matrix. Suppose that the eigenvectors of $\mathbf{A}$ form a complete set, i.e., span$(\mathbf{v}_1 \cdots \mathbf{v}_n) = \mathbb{R}^n$ (i.e., the set of eigenvectors $(\mathbf{v}_1, \ldots, \mathbf{v}_n)$ form a linearly independent set that spans $\mathbb{R}^n$). Prove the following result:

$$\mathbf{D} = \mathbf{V}^{-1} \mathbf{A} \mathbf{V}$$

where $\mathbf{V} = [\mathbf{v}_1 \cdots \mathbf{v}_n]$ is the matrix of eigenvectors and $\mathbf{D} = \text{diag}(\lambda_1, \ldots, \lambda_n)$ is a diagonal matrix whose diagonal elements are the eigenvalues of $\mathbf{A}$.

Question 9

Let $p(s)$ be a polynomial in $s \in \mathbb{C}$ of degree n. Furthermore, let $\mathbf{A} \in \mathbb{R}^{n \times n}$ be an arbitrary square matrix. Finally, let $\mathbf{T} \in \mathbb{R}^{n \times n}$ be a nonsingular square matrix. Prove that

$$p(\mathbf{T}^{-1} \mathbf{A} \mathbf{T}) = \mathbf{T}^{-1} p(\mathbf{A}) \mathbf{T}$$

Question 10

Prove the following properties of the state transition matrix $\mathbf{\Phi}(t, \tau)$:

(a) $\mathbf{\Phi}(t, \tau) = \mathbf{\Phi}(t, t_1) \mathbf{\Phi}(t_1, \tau)$

(b) $\mathbf{\Phi}(t,\tau) = \mathbf{\Phi}^{-1}(\tau,t)$

(c) $\dfrac{\partial \mathbf{\Phi}(t,\tau)}{\partial t} = \mathbf{A}(t)\mathbf{\Phi}(t,\tau)$

(d) $\dfrac{\partial \mathbf{\Phi}(t,\tau)}{\partial \tau} = -\mathbf{\Phi}(t,\tau)\mathbf{A}(\tau)$

(e) $\mathbf{\Phi}(t,\tau)$ is nonsingular $\forall\, t$ and τ

Question 11

Let $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$ be a linear time-invariant system with initial condition $\mathbf{x}(0) = \mathbf{x}_0$. The solution of this system can be written implicitly in integral form as

$$\mathbf{x}(t) = \mathbf{x}(0) + \int_0^t \mathbf{A}\mathbf{x}(\tau)d\tau$$

Suppose now that we use the following iterative procedure to solve the problem:

$$\mathbf{x}^{(k+1)}(t) = \mathbf{x}(0) + \int_0^t \mathbf{A}\mathbf{x}^{(k)}(\tau)d\tau$$

where $\mathbf{x}^{(k)}(t)$ is the solution of the k^{th} iteration ($k \geq 0$). Assuming that the zeroth iteration is $\mathbf{x}^{(0)} = \mathbf{x}_0$, prove the following result:

$$\lim_{k\to\infty} \mathbf{x}^{(k)}(t) = \mathbf{x}(t) = e^{\mathbf{A}t}\mathbf{x}_0$$

Repeat the procedure for $\mathbf{x}^{(0)} = \bar{\mathbf{x}} \neq \mathbf{x}_0$. How does your result differ in the second case as compared to the first case?