Question 1

Using the approach for calculus of variations developed in class (not the approach used in Kirk's book), derive the necessary conditions for optimality that minimize the integral

$$J = \int_{t_0}^{t_f} \mathcal{L}[x(t), \dot{x}(t), t]dt$$

for the following sets of boundary conditions:

- t_0 fixed, $x(t_0) = x_0$ fixed; t_f fixed, $x(t_f) = x_f$ fixed
- t_0 fixed, $x(t_0) = x_0$ fixed; t_f fixed, $x(t_f) = x_f$ free
- t_0 fixed, $x(t_0) = x_0$ fixed; t_f free, $x(t_f) = x_f$ fixed
- t_0 fixed, $x(t_0) = x_0$ fixed; t_f free, $x(t_f) = x_f$ free

Question 2

Prove the following theorem (known as the *fundamental lemma of variational calculus*). Given a continuous function $f(t)$ on the time interval $t \in [t_0, t_f]$ and that

$$\int_{t_0}^{t_f} f(t)\delta x(t) = 0$$

for all continuous functions $\delta x(t)$ on $t \in [t_0, t_f]$ such that $\delta x(t_0) = \delta x(t_f) = 0$. Then $\delta x(t)$ must be identically zero on $t \in [t_0, t_f]$.

Question 3 (Kirk 1998)

Determine the extremal solutions of the following two functionals:

(1) $J(x(t)) = \int_0^1 [x^2(t) - \dot{x}^2(t)]\, dt$, $x(0) = 1$, $x(1) = 1$

(2) $J(x(t)) = \int_0^2 [x^2(t) - 2\dot{x}(t)x(t) + \dot{x}^2(t)]\, dt$, $x(0) = 1$, $x(2) = -3$

Question 4

Determine the extremal curve $x(t)$ of the functional

$$J(x(t)) = \int_0^{t_f} \sqrt{1 + \dot{x}^2(t)}dt$$

with boundary conditions

$$\begin{aligned} x(0) &= 5 \\ x^2(t_f) + (t_f - 5)^2 - 4 &= 0 \end{aligned}$$

where t_f is free. Plot the extremal solution and give a geometric interpretation of the result.

Question 5

The *Brachistochrone Problem* is one of the earliest and most famous problems in the calculus of variations. The brachistochrone problem is stated as follows. Let $x(t)$ and $y(t)$ measure the horizontal and vertically downward components of position of a particle of mass m in an inertially fixed Cartesian coordinate system. Starting at the the point $(x(t_0), y(t_0)) = (0, 0)$, determine the path along which a particle of mass m must move under the influence of constant gravity g such that it reaches the point $x(t_f), y(t_f) = (1, 1)$ in minimum time. Assume in your solution that $t_0 = 0$.