Chapter 2

Kinematics

Question 2–1

A bug B crawls radially outward at constant speed v_0 from the center of a rotating disk as shown in Fig. P2-1. Knowing that the disk rotates about its center O with constant absolute angular velocity $\mathbf{\Omega}$ relative to the ground (where $\|\mathbf{\Omega}\| = \Omega$), determine the velocity and acceleration of the bug as viewed by an observer fixed to the ground.

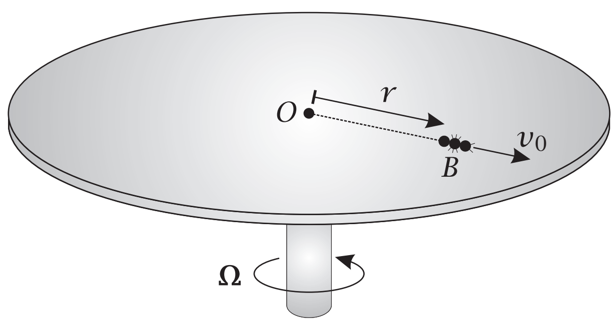

Figure P2-1

Solution to Question 2–1

For this problem it is convenient to choose a fixed reference frame $\mathcal{F}$ and a non-inertial reference frame $\mathcal{A}$ that is fixed in the disk. Corresponding to reference frame $\mathcal{F}$ we choose the following coordinate system:

	Origin at Point O	
$\mathbf{E}_x$	=	Along OB at Time $t = 0$
$\mathbf{E}_z$	=	Out of Page
$\mathbf{E}_y$	=	$\mathbf{E}_z \times \mathbf{E}_x$

Corresponding to the reference frame $\mathcal{A}$ that is fixed in the disk, we choose the following coordinate system

$$
\begin{array}{ccc}
 & \text{Origin at Point } O & \\
\mathbf{e}_x & = & \text{Along } OB \\
\mathbf{e}_z & = & \text{Out of Page } (= \mathbf{E}_z) \\
\mathbf{e}_y & = & \mathbf{e}_z \times \mathbf{e}_x
\end{array}
$$

The position of the bug is then resolved in the basis $\{\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z\}$ as

$$\mathbf{r} = r\mathbf{e}_x \tag{2.1}$$

Now, since the platform rotates about the $\mathbf{e}_z$-direction relative to the ground, the angular velocity of reference frame $\mathcal{A}$ in reference frame $\mathcal{F}$ is given as

$${}^{\mathcal{F}}\boldsymbol{\omega}^{\mathcal{A}} = \Omega\mathbf{e}_z \tag{2.2}$$

The velocity is found by applying the basic kinematic equation. This gives

$${}^{\mathcal{F}}\mathbf{v} = \frac{{}^{\mathcal{F}}d\mathbf{r}}{dt} = \frac{{}^{\mathcal{A}}d\mathbf{r}}{dt} + {}^{\mathcal{F}}\boldsymbol{\omega}^{\mathcal{A}} \times \mathbf{r} \tag{2.3}$$

Now we have

$$\frac{{}^{\mathcal{A}}d\mathbf{r}}{dt} = \dot{r}\mathbf{e}_x = v_0\mathbf{e}_x \tag{2.4}$$

$$
\begin{aligned}
{}^{\mathcal{F}}\boldsymbol{\omega}^{\mathcal{A}} \times \mathbf{r} &= \Omega\mathbf{e}_z \times r\mathbf{e}_x \\
&= \Omega r\mathbf{e}_y
\end{aligned} \tag{2.5}
$$

Adding Eqs. (2.4) and (2.5), we obtain the velocity of the bug in reference frame $\mathcal{F}$ as

$${}^{\mathcal{F}}\mathbf{v} = v_0\mathbf{e}_x + \Omega r\mathbf{e}_y \tag{2.6}$$

The acceleration is found by applying the basic kinematic equation to ${}^{\mathcal{F}}\mathbf{v}$. This gives

$${}^{\mathcal{F}}\mathbf{a} = \frac{{}^{\mathcal{F}}d}{dt}\left({}^{\mathcal{F}}\mathbf{v}\right) = \frac{{}^{\mathcal{A}}d}{dt}\left({}^{\mathcal{F}}\mathbf{v}\right) + {}^{\mathcal{F}}\boldsymbol{\omega}^{\mathcal{A}} \times {}^{\mathcal{F}}\mathbf{v} \tag{2.7}$$

Using ${}^{\mathcal{F}}\mathbf{v}$ from Eq. (2.6) and noting that v_0 and Ω are constant, we have that

$$
\begin{aligned}
\frac{{}^{\mathcal{A}}d}{dt}\left({}^{\mathcal{F}}\mathbf{v}\right) &= \Omega\dot{r}\mathbf{e}_y = \Omega v_0\mathbf{e}_y \\
{}^{\mathcal{F}}\boldsymbol{\omega}^{\mathcal{A}} \times {}^{\mathcal{F}}\mathbf{v} &= \Omega\mathbf{e}_z \times [v_0\mathbf{e}_x + r\Omega\mathbf{e}_y] \\
&= -\Omega^2 r\mathbf{e}_x + \Omega v_0\mathbf{e}_y
\end{aligned} \tag{2.8}
$$

Therefore, the acceleration in reference frame $\mathcal{F}$ is given as

$${}^{\mathcal{F}}\mathbf{a} = -\Omega^2 r\mathbf{e}_x + 2\Omega v_0\mathbf{e}_y \tag{2.9}$$

Question 2–2

A particle, denoted by P, slides on a circular table as shown in Fig. P2-2. The position of the particle is known in terms of the radius r measured from the center of the table at point O and the angle θ where θ is measured relative to the direction of OQ where Q is a point on the circumference of the table. Knowing that the table rotates with constant angular rate Ω, determine the velocity and acceleration of the particle as viewed by an observer in a fixed reference frame.

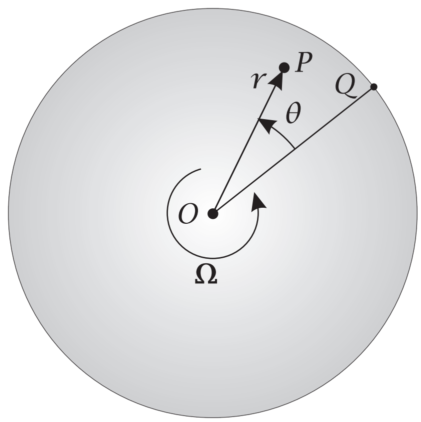

Figure P2-2

Solution to Question 2–2

For this problem it is convenient to define a fixed inertial reference frame $\mathcal{F}$ and two non-inertial reference frames $\mathcal{A}$ and $\mathcal{B}$. The first non-inertial reference frame $\mathcal{A}$ is fixed to the disk while the second non-inertial reference frame $\mathcal{B}$ is fixed to the direction of OP. Corresponding to the fixed inertial reference frame $\mathcal{F}$, we choose the following coordinate system:

	Origin at point O	
$\mathbf{E}_x$	$=$	Along Ox at $t = 0$
$\mathbf{E}_z$	$=$	Out of Page
$\mathbf{E}_y$	$=$	$\mathbf{E}_z \times \mathbf{E}_x$

Corresponding to non-inertial reference frame $\mathcal{A}$, we choose the following coordinate system:

	Origin at point O	
$\mathbf{e}_x$	$=$	Along OQ
$\mathbf{e}_z$	$=$	Out of Page ($= \mathbf{E}_z$)
$\mathbf{e}_y$	$=$	$\mathbf{e}_z \times \mathbf{e}_x$

Finally, corresponding to reference frame $\mathcal{B}$, we choose the following coordinate system:

$$\begin{array}{ccc} & \text{Origin at point } O & \\ \mathbf{e}_r & = & \text{Along } OP \\ \mathbf{e}_z & = & \text{Out of Page} \\ \mathbf{e}_\theta & = & \mathbf{e}_z \times \mathbf{e}_r \end{array}$$

Then, the position of the particle can be desribed in terms of the basis $\{\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_z\}$ as

$$\mathbf{r} = r\mathbf{e}_r. \tag{2.10}$$

Now, in order to compute the velocity of the particle, it is necessary to apply the basic kinematic equation. In this case since we are interested in motion as viewed by an observer in the fixed inertial reference frame $\mathcal{F}$, we need to determine the angular velocity of $\mathcal{B}$ in $\mathcal{F}$. First, since $\mathcal{A}$ rotates relative to $\mathcal{F}$ with angular velocity $\mathbf{\Omega}$, we have that

$$^{\mathcal{F}}\boldsymbol{\omega}^{\mathcal{A}} = \mathbf{\Omega} = \Omega\mathbf{e}_z \tag{2.11}$$

Next, since $\mathcal{B}$ rotates relative to $\mathcal{A}$ with angular rate $\dot{\theta}$ about the $\mathbf{e}_z$-direction, we have that

$$^{\mathcal{A}}\boldsymbol{\omega}^{\mathcal{B}} = \dot{\theta}\mathbf{e}_z \tag{2.12}$$

Then, applying the theorem of addition of angular velocities, we have that

$$^{\mathcal{F}}\boldsymbol{\omega}^{\mathcal{B}} = {}^{\mathcal{F}}\boldsymbol{\omega}^{\mathcal{A}} + {}^{\mathcal{A}}\boldsymbol{\omega}^{\mathcal{B}} = \Omega\mathbf{e}_z + \dot{\theta}\mathbf{e}_z = (\Omega + \dot{\theta})\mathbf{e}_z \tag{2.13}$$

The velocity in reference frame is then found by applying the rate of change transport theorem as

$$^{\mathcal{F}}\mathbf{v} = \frac{^{\mathcal{F}}d\mathbf{r}}{dt} = \frac{^{\mathcal{B}}d\mathbf{r}}{dt} + {}^{\mathcal{F}}\boldsymbol{\omega}^{\mathcal{B}} \times \mathbf{r} \tag{2.14}$$

Now we have

$$\frac{^{\mathcal{B}}d\mathbf{r}}{dt} = \dot{r}\mathbf{e}_r \tag{2.15}$$

$$^{\mathcal{F}}\boldsymbol{\omega}^{\mathcal{B}} \times \mathbf{r} = (\Omega + \dot{\theta})\mathbf{e}_z \times r\mathbf{e}_r = r(\Omega + \dot{\theta})\mathbf{e}_\theta \tag{2.16}$$

Adding Eqs. (2.15) and (2.16), we obtain the velocity of the particle in reference frame $\mathcal{F}$ as

$$^{\mathcal{F}}\mathbf{v} = \dot{r}\mathbf{e}_r + r(\Omega + \dot{\theta})\mathbf{e}_\theta \tag{2.17}$$

The acceleration is found by applying the rate of change transport theorem to$^{\mathcal{F}}\mathbf{v}$. This gives

$$^{\mathcal{F}}\mathbf{a} = \frac{^{\mathcal{F}}d}{dt}\left(^{\mathcal{F}}\mathbf{v}\right) = \frac{^{\mathcal{B}}d}{dt}\left(^{\mathcal{F}}\mathbf{v}\right) + {}^{\mathcal{F}}\boldsymbol{\omega}^{\mathcal{B}} \times {}^{\mathcal{F}}\mathbf{v} \tag{2.18}$$

Using ${}^{\mathcal{F}}\mathbf{v}$ from Eq. (2.17) and noting again that Ω is constant, we have

$$\begin{aligned} \frac{{}^{\mathcal{B}}d}{dt}\left({}^{\mathcal{F}}\mathbf{v}\right) &= \ddot{r}\mathbf{e}_r + \left[\dot{r}(\Omega + \dot{\theta}) + r\ddot{\theta}\right]\mathbf{e}_\theta && (2.19) \\ {}^{\mathcal{F}}\boldsymbol{\omega}^{\mathcal{B}} \times {}^{\mathcal{F}}\mathbf{v} &= (\Omega + \dot{\theta})\mathbf{e}_z \times \left[\dot{r}\mathbf{e}_r + r(\Omega + \dot{\theta})\mathbf{e}_\theta\right] \\ &= -r(\Omega + \dot{\theta})^2\mathbf{e}_r + \dot{r}(\Omega + \dot{\theta})\mathbf{e}_\theta && (2.20) \end{aligned}$$

Adding Eqs. (2.19) and (2.20), we obtain the acceleration of the particle in reference frame $\mathcal{F}$ as

$${}^{\mathcal{F}}\mathbf{a} = \left[\ddot{r} - r(\Omega + \dot{\theta})^2\right]\mathbf{e}_r + \left[r\ddot{\theta} + 2\dot{r}(\Omega + \dot{\theta})\right]\mathbf{e}_\theta \qquad (2.21)$$

Question 2–3

A collar slides along a rod as shown in Fig. P2-3. The rod is free to rotate about a hinge at the fixed point O. Simultaneously, the rod rotates about the vertical direction with constant angular velocity $\boldsymbol{\Omega}$ relative to the ground. Knowing that r describes the location of the collar along the rod, that θ is the angle measured from the vertical, and that $\Omega = \|\boldsymbol{\Omega}\|$, determine the velocity and acceleration of the collar as viewed by an observer fixed to the ground.

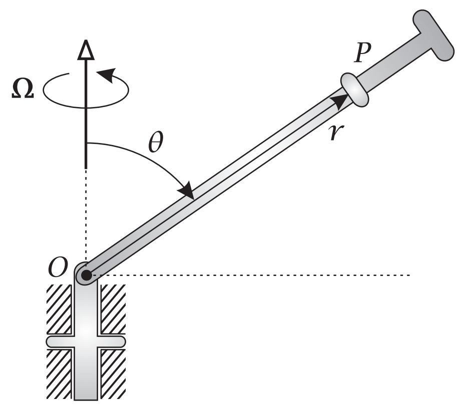

Figure P2-3

Solution to Question 2–3

First, let $\mathcal{F}$ be a fixed reference frame. Then, choose the following coordinate system fixed in reference frame $\mathcal{F}$:

	Origin at point O	
$\mathbf{E}_x$	=	Along $\boldsymbol{\Omega}$
$\mathbf{E}_z$	=	Orthogonal to Plane of Shaft and Arm at $t = 0$
$\mathbf{E}_y$	=	$\mathbf{E}_z \times \mathbf{E}_x$

Next, let $\mathcal{A}$ be a reference frame fixed to the vertical shaft. Then, choose the following coordinate system fixed in reference frame $\mathcal{A}$:

	Origin at point O	
$\mathbf{e}_x$	=	Along $\boldsymbol{\Omega}$
$\mathbf{e}_z$	=	Orthogonal to Plane of Shaft and Arm
$\mathbf{e}_y$	=	$\mathbf{e}_z \times \mathbf{e}_x$

Finally, let $\mathcal{B}$ be a reference frame fixed to the rod. Then, choose the following coordinate system fixed in reference frame $\mathcal{B}$:

$$
\begin{array}{ccc}
 & \text{Origin at point } O & \\
\mathbf{e}_r & = & \text{Along } OP \\
\mathbf{e}_z & = & \mathbf{u}_z \\
\mathbf{e}_\theta & = & \mathbf{e}_z \times \mathbf{e}_r
\end{array}
$$

The geometry of the bases $\{\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z\}$ and $\{\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_z\}$ is shown in Fig. 2-1. Using Fig. 2-1, the relationship between the basis $\{\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z\}$ and $\{\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_z\}$ is given as

$$
\begin{array}{rcl}
\mathbf{e}_x & = & \cos\theta\,\mathbf{e}_r - \sin\theta\,\mathbf{e}_\theta \\
\mathbf{e}_y & = & \sin\theta\,\mathbf{e}_r + \cos\theta\,\mathbf{e}_\theta
\end{array}
\tag{2.22}
$$

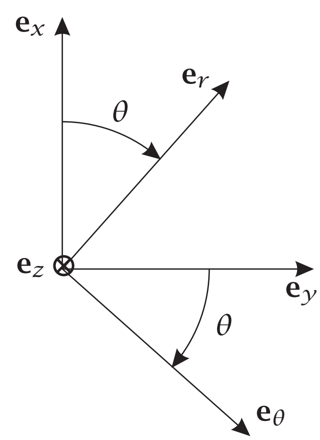

Figure 2-1 Geometry of Bases $\{\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z\}$ and $\{\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_z\}$ for Question 2–3.

The position of the particle can then be expressed in the basis $\{\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_z\}$ as

$$
\mathbf{r} = r\mathbf{e}_r \tag{2.23}
$$

Now, since $\{\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_z\}$ is fixed in reference frame $\mathcal{B}$, and we are interested in obtaining the velocity and acceleration as viewed by an observer fixed in the ground (i.e., reference frame $\mathcal{F}$), we need to obtain an expression for the angular velocity of reference frame $\mathcal{B}$ in reference frame $\mathcal{F}$. First, since reference frame $\mathcal{A}$ rotates relative to reference frame $\mathcal{F}$ with angular velocity $\boldsymbol{\Omega}$ and $\boldsymbol{\Omega}$ lies along the $\mathbf{e}_x$-direction, we have that

$$
{}^{\mathcal{F}}\boldsymbol{\omega}^{\mathcal{A}} = \boldsymbol{\Omega} = \Omega\mathbf{e}_x \tag{2.24}
$$

Next, since reference frame $\mathcal{B}$ rotates relative to reference frame $\mathcal{A}$ with angular rate $\dot{\theta}$ about the $\mathbf{e}_z$-direction. Therefore,

$$
{}^{\mathcal{A}}\boldsymbol{\omega}^{\mathcal{B}} = \dot{\theta}\mathbf{e}_z \tag{2.25}
$$

Then, using the angular velocity addition theorem, we have the angular velocity of reference frame $\mathcal{B}$ in reference frame $\mathcal{F}$ as

$$^{\mathcal{F}}\boldsymbol{\omega}^{\mathcal{B}} = {}^{\mathcal{F}}\boldsymbol{\omega}^{\mathcal{A}} + {}^{\mathcal{A}}\boldsymbol{\omega}^{\mathcal{B}} = \Omega\mathbf{e}_x + \dot{\theta}\mathbf{e}_z \tag{2.26}$$

Now, since we have determined that the position of the collar is expressed most conveniently in terms of the basis $\{\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_z\}$, it is also most convenient to express $^{\mathcal{F}}\boldsymbol{\omega}^{\mathcal{B}}$ in terms of the basis $\{\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_z\}$. In particular, substituting the expression for $\mathbf{e}_x$ from Eq. (2.22) into Eq. (2.26), we obtain $^{\mathcal{F}}\boldsymbol{\omega}^{\mathcal{B}}$ as

$$^{\mathcal{F}}\boldsymbol{\omega}^{\mathcal{B}} = \Omega(\cos\theta\,\mathbf{e}_r - \sin\theta\,\mathbf{e}_\theta) + \dot{\theta}\mathbf{e}_z = \Omega\cos\theta\,\mathbf{e}_r - \Omega\sin\theta\,\mathbf{e}_\theta + \dot{\theta}\mathbf{e}_z \tag{2.27}$$

The velocity in reference frame $\mathcal{F}$ is then found by applying the rate of change transport theorem between reference frames $\mathcal{B}$ and $\mathcal{F}$ as

$$^{\mathcal{F}}\mathbf{v} = \frac{^{\mathcal{F}}d\mathbf{r}}{dt} = \frac{^{\mathcal{B}}d\mathbf{r}}{dt} + {}^{\mathcal{F}}\boldsymbol{\omega}^{\mathcal{B}} \times \mathbf{r} \tag{2.28}$$

Now we have that

$$\frac{^{\mathcal{B}}d\mathbf{r}}{dt} = \dot{r}\mathbf{e}_r \tag{2.29}$$

$$\begin{aligned} ^{\mathcal{F}}\boldsymbol{\omega}^{\mathcal{B}} \times \mathbf{r} &= (\Omega\cos\theta\,\mathbf{e}_r - \Omega\sin\theta\,\mathbf{e}_\theta + \dot{\theta}\mathbf{e}_z) \times r\mathbf{e}_r \\ &= \Omega r\sin\theta\,\mathbf{E}_z + r\dot{\theta}\mathbf{e}_\theta \end{aligned} \tag{2.30}$$

Adding Eq. (2.29) and Eq. (2.30), we obtain the velocity of the collar in reference frame $\mathcal{F}$ as

$$^{\mathcal{F}}\mathbf{v} = \dot{r}\mathbf{e}_r + r\dot{\theta}\mathbf{e}_\theta + r\Omega\sin\theta\,\mathbf{e}_z \tag{2.31}$$

The acceleration of the collar is then obtained by applying the rate of change transport theorem to $^{\mathcal{F}}\mathbf{v}$ between reference frames $\mathcal{B}$ and $\mathcal{F}$ as

$$^{\mathcal{F}}\mathbf{a} = \frac{^{\mathcal{F}}d}{dt}\left(^{\mathcal{F}}\mathbf{v}\right) = \frac{^{\mathcal{B}}d}{dt}\left(^{\mathcal{F}}\mathbf{v}\right) + {}^{\mathcal{F}}\boldsymbol{\omega}^{\mathcal{B}} \times {}^{\mathcal{F}}\mathbf{v} \tag{2.32}$$

Now we have

$$\frac{^{\mathcal{B}}d}{dt}\left(^{\mathcal{F}}\mathbf{v}\right) = \ddot{r}\mathbf{e}_r + (\dot{r}\dot{\theta} + r\ddot{\theta})\mathbf{e}_\theta + \left[\Omega(\dot{r}\sin\theta + r\dot{\theta}\cos\theta)\right]\mathbf{e}_z \tag{2.33}$$

$$\begin{aligned} ^{\mathcal{F}}\boldsymbol{\omega}^{\mathcal{B}} \times {}^{\mathcal{F}}\mathbf{v} &= (\Omega\cos\theta\,\mathbf{e}_r - \Omega\sin\theta\,\mathbf{e}_\theta + \dot{\theta}\mathbf{e}_z) \times (\dot{r}\mathbf{e}_r + r\dot{\theta}\mathbf{e}_\theta + r\Omega\sin\theta\,\mathbf{e}_z) \\ &= r\Omega\dot{\theta}\cos\theta\,\mathbf{e}_z - r\Omega^2\cos\theta\,\sin\theta\,\mathbf{e}_\theta + \dot{r}\Omega\sin\theta\,\mathbf{e}_z - r\Omega^2\sin^2\theta\mathbf{e}_r \\ &\quad + \dot{r}\dot{\theta}\mathbf{e}_\theta - r\dot{\theta}^2\mathbf{e}_r \\ &= -(r\dot{\theta}^2 + r\Omega^2\sin^2\theta)\mathbf{e}_r + (\dot{r}\dot{\theta} - r\Omega^2\cos\theta\,\sin\theta)\mathbf{e}_\theta \\ &\quad + (r\Omega\dot{\theta}\cos\theta + \dot{r}\Omega\sin\theta)\mathbf{e}_z \end{aligned} \tag{2.34}$$

Adding Eqs. (2.33) and (2.34), we obtain the acceleration of the collar in reference frame $\mathcal{F}$ as

$$\begin{aligned} ^{\mathcal{F}}\mathbf{a} = (\ddot{r} - r\dot{\theta}^2 - r\Omega^2\sin^2\theta)\mathbf{e}_r + (2\dot{r}\dot{\theta} + r\ddot{\theta} - r\Omega^2\cos\theta\,\sin\theta)\mathbf{e}_\theta \\ +2\Omega(\dot{r}\sin\theta + r\dot{\theta}\cos\theta)\mathbf{e}_z \end{aligned} \tag{2.35}$$

Question 2–4

A particle slides along a track in the form of a parabola $y = x^2/a$ as shown in Fig. P2-4. The parabola rotates about the vertical with a constant angular velocity $\mathbf{\Omega}$ relative to a fixed reference frame (where $\Omega = \|\mathbf{\Omega}\|$). Determine the velocity and acceleration of the particle as viewed by an observer in a fixed reference frame.

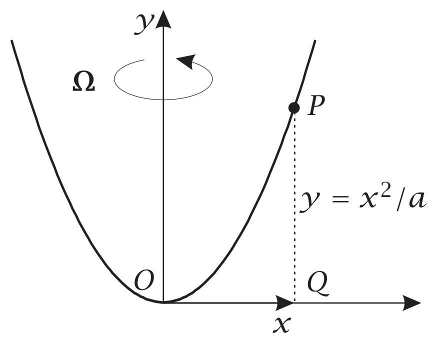

Figure P2-4

Solution to Question 2–4

For this problem it is convenient to define a fixed inertial reference frame $\mathcal{F}$ and a non-inertial reference frame $\mathcal{A}$. Corresponding to reference frame $\mathcal{F}$, we choose the following coordinate system:

	Origin at Point O	
$\mathbf{E}_x$	=	Along OQ When $t = 0$
$\mathbf{E}_y$	=	Along Oy When $t = 0$
$\mathbf{E}_z$	=	$\mathbf{E}_x \times \mathbf{E}_y$

Furthermore, corresponding to reference frame $\mathcal{A}$, we choose the following coordinate system:

	Origin at Point O	
$\mathbf{e}_x$	=	Along OQ
$\mathbf{e}_y$	=	Along Oy
$\mathbf{e}_z$	=	$\mathbf{e}_x \times \mathbf{e}_y$

The position of the particle is then given in terms of the basis $\{\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z\}$ as

$$\mathbf{r} = x\mathbf{e}_x + y\mathbf{e}_y = x\mathbf{e}_x + (x^2/a)\mathbf{e}_y \tag{2.36}$$

Furthermore, since the parabola spins about the ey-direction, the angular velocity of reference frame $\mathcal{A}$ in reference frame $\mathcal{F}$ is given as

$${}^{\mathcal{F}}\boldsymbol{\omega}^{\mathcal{A}} = \mathbf{\Omega} = \Omega\mathbf{e}_y \tag{2.37}$$

The velocity in reference frame $\mathcal{F}$ is then found using the rate of change transport theorem as

$$^{\mathcal{F}}\mathbf{v} = \frac{^{\mathcal{F}}d\mathbf{r}}{dt} = \frac{^{\mathcal{A}}d\mathbf{r}}{dt} + {}^{\mathcal{F}}\boldsymbol{\omega}^{\mathcal{A}} \times \mathbf{r} \tag{2.38}$$

Using $\mathbf{r}$ from Eq. (2.36) and $^{\mathcal{F}}\boldsymbol{\omega}^{\mathcal{A}}$ from Eq. (2.37), we have

$$\frac{^{\mathcal{A}}d\mathbf{r}}{dt} = \dot{x}\mathbf{e}_x + (2x\dot{x}/a)\mathbf{e}_y \tag{2.39}$$

$$^{\mathcal{F}}\boldsymbol{\omega}^{\mathcal{A}} \times \mathbf{r} = \Omega\mathbf{e}_y \times (x\mathbf{e}_x + (x^2/a)\mathbf{e}_y) = -\Omega x\mathbf{e}_z \tag{2.40}$$

Adding Eqs. (2.39) and (2.40), we obtain $^{\mathcal{F}}\mathbf{v}$ as

$$^{\mathcal{F}}\mathbf{v} = \dot{x}\mathbf{e}_x + (2x\dot{x}/a)\mathbf{e}_y - \Omega x\mathbf{e}_z \tag{2.41}$$

The acceleration in reference frame $\mathcal{F}$ is found by applying the rate of change transport theorem to $^{\mathcal{F}}\mathbf{v}$ as

$$^{\mathcal{F}}\mathbf{a} = \frac{^{\mathcal{F}}d}{dt}\left(^{\mathcal{F}}\mathbf{v}\right) = \frac{^{\mathcal{A}}d}{dt}\left(^{\mathcal{F}}\mathbf{v}\right) + {}^{\mathcal{F}}\boldsymbol{\omega}^{\mathcal{A}} \times {}^{\mathcal{F}}\mathbf{v} \tag{2.42}$$

Using $^{\mathcal{F}}\mathbf{v}$ from Eq. (2.41) and $^{\mathcal{F}}\boldsymbol{\omega}^{\mathcal{A}}$ from Eq. (2.37), we have

$$\frac{^{\mathcal{A}}d}{dt}\left(^{\mathcal{F}}\mathbf{v}\right) = \ddot{x}\mathbf{e}_x + \left[2(\dot{x}^2 + x\ddot{x})/a\right]\mathbf{e}_y - \Omega\dot{x}\mathbf{e}_z \tag{2.43}$$

$$^{\mathcal{F}}\boldsymbol{\omega}^{\mathcal{A}} \times {}^{\mathcal{F}}\mathbf{v} = \Omega\mathbf{e}_y \times (\dot{x}\mathbf{e}_x + (2x\dot{x}/a)\mathbf{e}_y - \Omega x\mathbf{e}_z) = -\Omega\dot{x}\mathbf{e}_z - \Omega^2 x\mathbf{e}_x \tag{2.44}$$

Adding Eq. (2.43) and (2.44), we obtain $^{\mathcal{F}}\mathbf{a}$ as

$$^{\mathcal{F}}\mathbf{a} = (\ddot{x} - \Omega^2 x)\mathbf{e}_x + \left[2(\dot{x}^2 + x\ddot{x})/a\right]\mathbf{e}_y - 2\Omega\dot{x}\mathbf{e}_z \tag{2.45}$$

Question 2–5

A satellite is in motion over the Earth as shown in Fig. P2-5. The Earth is modeled as a sphere of radius R that rotates with constant angular velocity $\boldsymbol{\Omega}$ in a direction $\mathbf{e}_z$ where $\mathbf{e}_z$ lies along a radial line that lies in the direction from the center of the Earth at point O to the North Pole of the Earth at point N. Furthermore, the center of the Earth is assumed to be an absolutely *fixed point*. The position of the satellite is known in terms of an *Earth-centered Earth-fixed* Cartesian coordinate system whose right-handed basis $\{\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z\}$ is defined as follows:

- The direction $\mathbf{e}_x$ lies orthogonal to $\mathbf{e}_z$ in the equatorial plane of the Earth along the line from O to P where P lies at the intersection of the equator with the great circle called the *Prime Meridian*

- The direction $\mathbf{e}_y$ lies orthogonal to both $\mathbf{e}_x$ and $\mathbf{e}_z$ in the equatorial plane of the Earth such that $\mathbf{e}_y = \mathbf{e}_z \times \mathbf{e}_x$

Using the basis $\{\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z\}$ to express all quantities, determine the velocity and acceleration of the spacecraft (a) as viewed by an observer fixed to the Earth and (b) as viewed by an observer in a fixed inertial reference frame.

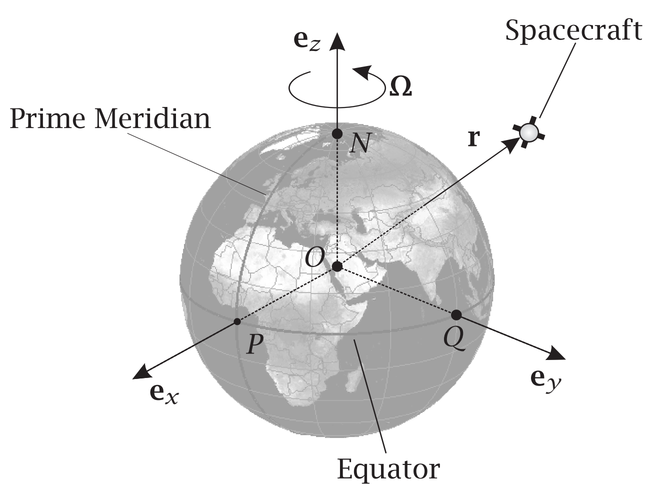

Figure P2-5

Solution to Question 2–5

First, let $\mathcal{F}$ be a fixed inertial reference frame. Next, let $\mathcal{A}$ be a reference frame that is fixed in the planet. Corresponding to reference frame $\mathcal{A}$, we choose the following coordinate system:

	Origin at point O	
$\mathbf{e}_x$	=	Along OP
$\mathbf{e}_z$	=	Along ON
$\mathbf{e}_y$	=	$\mathbf{e}_z \times \mathbf{e}_x$ (= Along OQ)

The position of the spacecraft is then given in terms of the basis $\{\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z\}$ as

$$\mathbf{r} = x\mathbf{e}_x + y\mathbf{e}_y + z\mathbf{e}_z \tag{2.46}$$

Now, since the planet rotates with constant angular velocity $\boldsymbol{\Omega}$ about the ON-direction relative to reference frame $\mathcal{F}$, we have that

$$^{\mathcal{F}}\boldsymbol{\omega}^{\mathcal{A}} = \Omega\mathbf{e}_z \tag{2.47}$$

The velocity of the spacecraft is then found by applying the rate of change transport theorem as

$$^{\mathcal{F}}\mathbf{v} = \frac{^{\mathcal{F}}d\mathbf{r}}{dt} = \frac{^{\mathcal{A}}d\mathbf{r}}{dt} + {^{\mathcal{F}}\boldsymbol{\omega}^{\mathcal{A}}} \times \mathbf{r} \tag{2.48}$$

Now we have

$$\frac{^{\mathcal{A}}d\mathbf{r}}{dt} = \dot{x}\mathbf{e}_x + \dot{y}\mathbf{e}_y + \dot{z}\mathbf{e}_z \tag{2.49}$$

$$\begin{aligned} {^{\mathcal{F}}\boldsymbol{\omega}^{\mathcal{A}}} \times \mathbf{r} &= \Omega\mathbf{e}_z \times (x\mathbf{e}_x + y\mathbf{e}_y + z\mathbf{e}_z) \\ &= \Omega x\mathbf{e}_y - \Omega y\mathbf{e}_x \end{aligned} \tag{2.50}$$

Adding Eqs. (2.49) and (2.50), we obtain $^{\mathcal{F}}\mathbf{v}$ as

$$^{\mathcal{F}}\mathbf{v} = (\dot{x} - \Omega y)\mathbf{e}_x + (\dot{y} + \Omega x)\mathbf{e}_y + \dot{z}\mathbf{e}_z \tag{2.51}$$

Next, the acceleration of the spacecraft in reference frame $\mathcal{F}$ is found by applying the rate of change transport theorem to $^{\mathcal{F}}\mathbf{v}$ as

$$^{\mathcal{F}}\mathbf{a} = \frac{^{\mathcal{F}}d}{dt}\left(^{\mathcal{F}}\mathbf{v}\right) = \frac{^{\mathcal{A}}d}{dt}\left(^{\mathcal{F}}\mathbf{v}\right) + {^{\mathcal{F}}\boldsymbol{\omega}^{\mathcal{A}}} \times {^{\mathcal{F}}\mathbf{v}} \tag{2.52}$$

Now we have

$$\frac{^{\mathcal{A}}d}{dt}\left(^{\mathcal{F}}\mathbf{v}\right) = (\ddot{x} - \Omega\dot{y})\mathbf{e}_x + (\ddot{y} + \Omega\dot{x})\mathbf{e}_y + \ddot{z}\mathbf{e}_z \tag{2.53}$$

$$\begin{aligned} {^{\mathcal{F}}\boldsymbol{\omega}^{\mathcal{A}}} \times {^{\mathcal{F}}\mathbf{v}} &= \Omega\mathbf{e}_z \times [(\dot{x} - \Omega y)\mathbf{e}_x + (\dot{y} + \Omega x)\mathbf{e}_y + \dot{z}\mathbf{e}_z] \\ &= \Omega(\dot{x} - \Omega y)\mathbf{e}_y - \Omega(\dot{y} + \Omega x)\mathbf{e}_x \end{aligned} \tag{2.54}$$

Adding Eqs. (2.53) and (2.54), we obtain $^{\mathcal{F}}\mathbf{a}$ as

$$^{\mathcal{F}}\mathbf{a} = (\ddot{x} - 2\Omega\dot{y} - \Omega^2 x)\mathbf{e}_x + (\ddot{y} + 2\Omega\dot{x} - \Omega^2 y)\mathbf{e}_y + \ddot{z}\mathbf{e}_z \tag{2.55}$$

Question 2–8

A bead slides along a fixed circular helix of radius R and helical inclination angle ϕ as shown in Fig. P2-8. Knowing that the angle θ measures the position of the bead and is equal to zero when the bead is at the base of the helix, determine the following quantities relative to an observer fixed to the helix: (a) the arclength parameter s as a function of the angle θ, (b) the intrinsic basis $\{\mathbf{e}_t, \mathbf{e}_n, \mathbf{e}_b\}$ and the curvature of the trajectory as a function of the angle θ, and (c) the position, velocity, and acceleration of the particle in terms of the intrinsic basis $\{\mathbf{e}_t, \mathbf{e}_n, \mathbf{e}_b\}$.

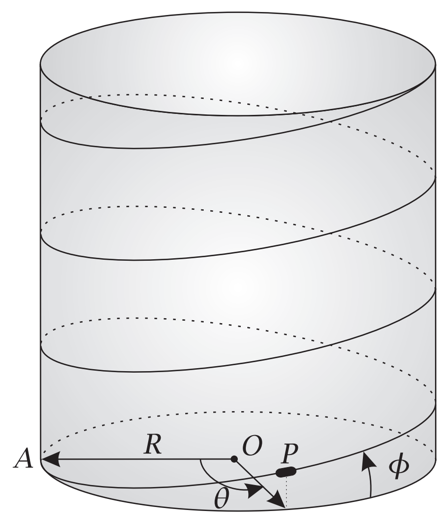

Figure P2-8

Solution to Question 2–8

Let $\mathcal{F}$ be a reference frame fixed to the helix. Then, choose the following coordinate system fixed in reference frame $\mathcal{F}$:

	Origin at O	
$\mathbf{E}_x$	$=$	Along OA
$\mathbf{E}_z$	$=$	Out of page
$\mathbf{E}_y$	$=$	$\mathbf{E}_z \times \mathbf{E}_x$

Next, let $\mathcal{A}$ be a reference frame that rotates with the projection of the position of particle into the $\left\{\mathbf{E}_x, \mathbf{E}_y\right\}$-plane. Corresponding to $\mathcal{A}$, we choose the following coordinate system to describe the motion of the particle:

$$
\begin{array}{lcl}
 & \text{Origin at } O & \\
\mathbf{e}_r & = & \text{Along } O \text{ to projection of } P \text{ into } \left\{\mathbf{E}_x, \mathbf{E}_y\right\} \text{ plane} \\
\mathbf{e}_z & = & \mathbf{E}_z \\
\mathbf{e}_\theta & = & \mathbf{e}_z \times \mathbf{e}_r
\end{array}
$$

Now, since ϕ is the angle formed by the helix with the horizontal, we have from the geometry that

$$z = R\theta \tan\phi \tag{2.56}$$

Then the position of the bead can be written as

$$\mathbf{r} = R\mathbf{e}_r + \tan\phi R\theta\mathbf{e}_z = R\mathbf{e}_r + R\theta\tan\phi\mathbf{e}_z \tag{2.57}$$

Furthermore, the angular velocity of reference frame $\mathcal{A}$ in reference frame $\mathcal{F}$ is given as

$${}^{\mathcal{F}}\boldsymbol{\omega}^{\mathcal{A}} = \dot{\theta}\mathbf{e}_z \tag{2.58}$$

Then, applying the rate of change transport theorem to $\mathbf{r}$ between reference frames $\mathcal{A}$ and $\mathcal{F}$, we have

$${}^{\mathcal{F}}\mathbf{v} = \frac{{}^{\mathcal{F}}d\mathbf{r}}{dt} = \frac{{}^{\mathcal{A}}d\mathbf{r}}{dt} + {}^{\mathcal{F}}\boldsymbol{\omega}^{\mathcal{A}} \times \mathbf{r} \tag{2.59}$$

where

$$\frac{{}^{\mathcal{A}}d\mathbf{r}}{dt} = R\dot{\theta}\tan\phi\mathbf{e}_z \tag{2.60}$$

$${}^{\mathcal{F}}\boldsymbol{\omega}^{\mathcal{A}} \times \mathbf{r} = \dot{\theta}\mathbf{e}_z \times (R\mathbf{e}_r + R\theta\tan\phi\mathbf{e}_z) = R\dot{\theta}\mathbf{e}_\theta \tag{2.61}$$

Adding Eqs. (2.60) and (2.61), we obtain

$${}^{\mathcal{F}}\mathbf{v} = R\dot{\theta}\mathbf{e}_\theta + R\dot{\theta}\tan\phi\mathbf{e}_z \tag{2.62}$$

The speed in reference frame $\mathcal{F}$ is then given as

$${}^{\mathcal{F}}v = \|{}^{\mathcal{F}}\mathbf{v}\| = R\dot{\theta}\sqrt{1+\tan^2\phi} = R\dot{\theta}\sec\phi \equiv \frac{d}{dt}\left({}^{\mathcal{F}}s\right) \tag{2.63}$$

Consequently,

$${}^{\mathcal{F}}ds = R\sec\phi d\theta \tag{2.64}$$

Integrating both sides of Eq. (2.64), we obtain

$$\int_{{}^{\mathcal{F}}s_0}^{{}^{\mathcal{F}}s} ds = \int_{\theta_0}^{\theta} R\sec\phi d\theta \tag{2.65}$$

We then obtain

$${}^{\mathcal{F}}s - {}^{\mathcal{F}}s_0 = R(\theta - \theta_0)\sec\phi \tag{2.66}$$

Solving Eq. (2.66) for s, the arclength is given as

$${}^{\mathcal{F}}s = {}^{\mathcal{F}}s_0 + R(\theta - \theta_0)\sec\phi \tag{2.67}$$

Intrinsic Basis and Curvature of Trajectory

The intrinsic basis is obtained as follows. First, the tangent vector $\mathbf{e}_t$ is given as

$$\mathbf{e}_t = \frac{^{\mathcal{F}}\mathbf{v}}{^{\mathcal{F}}v} \tag{2.68}$$

Substituting the expressions for $^{\mathcal{F}}\mathbf{v}$ and $^{\mathcal{F}}v$ from part (a) into Eq. (2.68), we obtain

$$\mathbf{e}_t = \frac{R\dot{\theta}\mathbf{e}_\theta + R\dot{\theta}\tan\phi\mathbf{e}_z}{R\dot{\theta}\sec\phi} = \cos\phi\mathbf{e}_\theta + \sin\phi\mathbf{e}_z \tag{2.69}$$

Next, we have

$$\frac{^{\mathcal{F}}d\mathbf{e}_t}{dt} = \frac{^{\mathcal{A}}d\mathbf{e}_t}{dt} + {}^{\mathcal{F}}\boldsymbol{\omega}^{\mathcal{A}} \times \mathbf{e}_t \tag{2.70}$$

where

$$\begin{aligned} \frac{^{\mathcal{A}}d\mathbf{e}_t}{dt} &= \mathbf{0} \\ {}^{\mathcal{F}}\boldsymbol{\omega}^{\mathcal{A}} \times \mathbf{e}_t &= \dot{\theta}\mathbf{e}_z \times (\cos\phi\mathbf{e}_\theta + \sin\phi\mathbf{e}_z) = -\dot{\theta}\cos\phi\mathbf{e}_r \end{aligned} \tag{2.71}$$

Therefore,

$$\frac{^{\mathcal{F}}d\mathbf{e}_t}{dt} = -\dot{\theta}\cos\phi\mathbf{e}_r \tag{2.72}$$

The principle unit normal is then given as

$$\mathbf{e}_n = \frac{^{\mathcal{F}}d\mathbf{e}_t/dt}{\|^{\mathcal{F}}d\mathbf{e}_t/dt\|} = -\mathbf{e}_r \tag{2.73}$$

Finally, the principle unit bi-normal vector is given as

$$\mathbf{e}_b = \mathbf{e}_t \times \mathbf{e}_n = (\cos\phi\mathbf{e}_\theta + \sin\phi\mathbf{e}_z) \times (-\mathbf{e}_r) = -\sin\phi\mathbf{e}_\theta + \cos\phi\mathbf{e}_z \tag{2.74}$$

Position, Velocity, and Acceleration of Bead

First, we can solve for the basis $\{\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_z\}$ in terms of $\{\mathbf{e}_t, \mathbf{e}_n, \mathbf{e}_b\}$ by using Eqs. (2.69), (2.73), and (2.74). First, from Eq. (2.73), we have

$$\mathbf{e}_r = -\mathbf{e}_n \tag{2.75}$$

Next, restating Eqs. (2.69) and (2.74), we have

$$\mathbf{e}_t = \cos\phi\mathbf{e}_\theta + \sin\phi\mathbf{e}_z \tag{2.76}$$

$$\mathbf{e}_b = -\sin\phi\mathbf{e}_\theta + \cos\phi\mathbf{e}_z \tag{2.77}$$

Solving Eqs. (2.76) and (2.77) simultaneously for $\mathbf{e}_\theta$ and $\mathbf{e}_z$, we obtain

$$\mathbf{e}_\theta = \sin\phi\mathbf{e}_t + \cos\phi\mathbf{e}_b \tag{2.78}$$

$$\mathbf{e}_z = \cos\phi\mathbf{e}_t - \sin\phi\mathbf{e}_b \tag{2.79}$$

Then, substituting the expressions for $\mathbf{e}_r$ and $\mathbf{e}_z$ from Eqs. (2.75) and (2.79) into Eq. (2.57), we have the position in terms of $\{\mathbf{e}_t, \mathbf{e}_n, \mathbf{e}_b\}$ as

$$\mathbf{r} = -R\mathbf{e}_n + R\theta \tan\phi(\cos\phi\mathbf{e}_t - \sin\phi\mathbf{e}_b) \tag{2.80}$$

Next, the velocity in reference frame $\mathcal{F}$ is given in terms of $\{\mathbf{e}_t, \mathbf{e}_n, \mathbf{e}_b\}$ as

$$^{\mathcal{F}}\mathbf{v} = {}^{\mathcal{F}}v\mathbf{e}_t \tag{2.81}$$

Substituting the expression for $^{\mathcal{F}}v$ from Eq. (2.63) into Eq. (2.81), we have

$$^{\mathcal{F}}\mathbf{v} = R\dot{\theta}\sec\phi\mathbf{e}_t \tag{2.82}$$

Finally, the acceleration in reference frame $\mathcal{F}$ is given as

$$^{\mathcal{F}}\mathbf{a} = \frac{d}{dt}\left({}^{\mathcal{F}}v\right)\mathbf{e}_t + {}^{\mathcal{F}}v\left\|\frac{^{\mathcal{F}}d\mathbf{e}_t}{dt}\right\|\mathbf{e}_n \tag{2.83}$$

Computing the rate of change of $^{\mathcal{F}}v$ using the expression for $^{\mathcal{F}}v$ from Eq. (2.63), we have

$$\frac{d}{dt}\left({}^{\mathcal{F}}v\right) = R\ddot{\theta}\sec\phi \tag{2.84}$$

Therefore,

$$^{\mathcal{F}}\mathbf{a} = R\ddot{\theta}\sec\phi\mathbf{e}_t + R\dot{\theta}\sec\phi\dot{\theta}\cos\phi\mathbf{e}_n = R\ddot{\theta}\sec\phi\mathbf{e}_t + R\dot{\theta}^2\mathbf{e}_n \tag{2.85}$$

Question 2–9

Arm AB is hinged at points A and B to collars that slide along vertical and horizontal shafts, respectively, as shown in Fig. P2-9. The vertical shaft rotates with angular velocity $\mathbf{\Omega}$ relative to a fixed reference frame (where $\Omega = \|\mathbf{\Omega}\|$) and point B moves with constant velocity $\mathbf{v}_0$ relative to the horizontal shaft. Knowing that point P is located at the center of the arm and the angle θ describes the orientation of the arm with respect to the vertical shaft, determine the velocity and acceleration of point P as viewed by an observer fixed to the ground. In simplifying your answers, find an expression for $\dot{\theta}$ in terms of v_0 and l and express your answers in terms of only l, Ω, $\dot{\Omega}$, θ, and v_0.

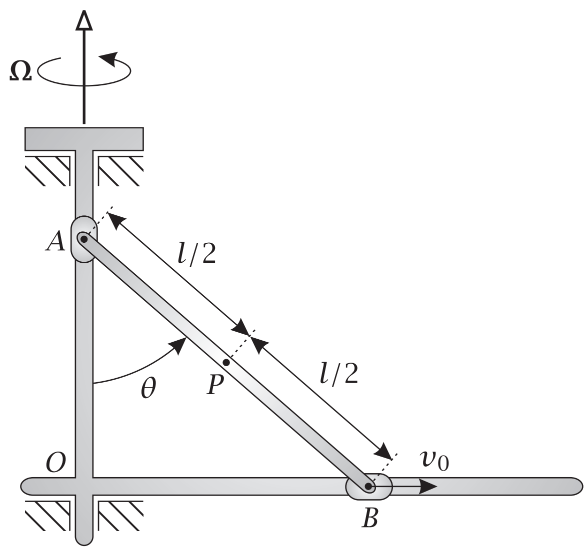

Figure P2-9

Solution to Question 2–9

Let $\mathcal{F}$ be the ground. Then choose the following coordinate system fixed in reference frame $\mathcal{F}$:

	Origin at O	
$\mathbf{E}_x$	=	Along OB when $t = 0$
$\mathbf{E}_y$	=	Along OA
$\mathbf{E}_z$	=	$\mathbf{E}_x \times \mathbf{E}_y$

Next, let $\mathcal{A}$ be the L-shaped assembly. Then choose the following coordinate system fixed in $\mathcal{A}$:

$$
\begin{array}{ccc}
 & \text{Origin at } O & \\
\mathbf{e}_x & = & \text{Along } OB \\
\mathbf{e}_y & = & \text{Along } OA \\
\mathbf{e}_z & = & \mathbf{e}_x \times \mathbf{e}_y
\end{array}
$$

Finally, let $\mathcal{B}$ be the rod. Then choose the following coordinate system fixed in $\mathcal{B}$:

$$
\begin{array}{ccc}
 & \text{Origin at } A & \\
\mathbf{u}_r & = & \text{Along } AB \\
\mathbf{u}_z & = & \mathbf{e}_z \\
\mathbf{u}_\theta & = & \mathbf{u}_z \times \mathbf{u}_r
\end{array}
$$

From the geometry of the coordinate systems, we have

$$
\begin{array}{rcl}
\mathbf{e}_x & = & \sin\theta\,\mathbf{u}_r + \cos\theta\,\mathbf{u}_\theta \\
\mathbf{e}_y & = & -\cos\theta\,\mathbf{u}_r + \sin\theta\,\mathbf{u}_\theta
\end{array}
\tag{2.86}
$$

Next, because we must measure all distances from point O (because point O is fixed to the ground and we want all rates of change as viewed by an observer fixed to the ground), the position of the center of the rod is given as

$$
\mathbf{r}_{P/O} = \mathbf{r}_{A/O} + \mathbf{r}_{P/A} \equiv \mathbf{r}
\tag{2.87}
$$

Using the coordinates systems defined for this problem, we have

$$
\begin{array}{rcl}
\mathbf{r}_{A/O} & = & l\cos\theta\,\mathbf{e}_y \\
\mathbf{r}_{P/A} & = & \frac{l}{2}\mathbf{u}_r
\end{array}
\tag{2.88}
$$

Consequently,

$$
\mathbf{r}_{P/O} = l\cos\theta\,\mathbf{e}_y + \frac{l}{2}\mathbf{u}_r
\tag{2.89}
$$

Because $\mathbf{r}_{A/O}$ is expressed in the basis $\{\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z\}$ while $\mathbf{r}_{P/A}$ is expressed in the basis $\{\mathbf{u}_r, \mathbf{u}_\theta, \mathbf{u}_z\}$, it is convenient to differentiate each piece of the vector $\mathbf{r}_{P/O}$ separately. First, the velocity of point A relative to point O as viewed by an observer fixed to the ground is obtained by applying the transport theorem from $\mathcal{A}$ to $\mathcal{F}$ as

$$
{}^{\mathcal{F}}\mathbf{v}_{A/O} = \frac{{}^{\mathcal{F}}d}{dt}\left(\mathbf{r}_{A/O}\right) = \frac{{}^{\mathcal{A}}d}{dt}\left(\mathbf{r}_{A/O}\right) + {}^{\mathcal{F}}\boldsymbol{\omega}^{\mathcal{A}} \times \mathbf{r}_{A/O}
\tag{2.90}
$$

First, we have

$$
{}^{\mathcal{F}}\boldsymbol{\omega}^{\mathcal{A}} = \boldsymbol{\Omega} = \Omega\mathbf{e}_y
\tag{2.91}
$$

Next,

$$
\begin{array}{rcl}
\frac{{}^{\mathcal{A}}d}{dt}\left(\mathbf{r}_{A/O}\right) & = & -l\dot{\theta}\sin\theta\,\mathbf{e}_y \\
{}^{\mathcal{F}}\boldsymbol{\omega}^{\mathcal{A}} \times \mathbf{r}_{A/O} & = & \Omega\mathbf{e}_y \times l\cos\theta\,\mathbf{e}_y = \mathbf{0}
\end{array}
\tag{2.92}
$$

Consequently,

$$^{\mathcal{F}}\mathbf{v}_{A/O} = -l\dot{\theta}\sin\theta\,\mathbf{e}_y \tag{2.93}$$

The acceleration of point A relative to point O as viewed by an observer fixed to the ground is then given as

$$^{\mathcal{F}}\mathbf{a}_{A/O} = \frac{^{\mathcal{F}}d}{dt}\left(^{\mathcal{F}}\mathbf{v}_{A/O}\right) = \frac{^{\mathcal{A}}d}{dt}\left(^{\mathcal{F}}\mathbf{v}_{A/O}\right) + {}^{\mathcal{F}}\boldsymbol{\omega}^{\mathcal{A}} \times {}^{\mathcal{F}}\mathbf{v}_{A/O} \tag{2.94}$$

Now we have

$$\begin{aligned} \frac{^{\mathcal{A}}d}{dt}(^{\mathcal{F}}\mathbf{v}_{A/O}) &= -l(\ddot{\theta}\sin\theta + \dot{\theta}^2\cos\theta)\mathbf{e}_y \\ ^{\mathcal{F}}\boldsymbol{\omega}^{\mathcal{A}} \times {}^{\mathcal{F}}\mathbf{v}_{A/O} &= \Omega\mathbf{e}_y \times (-l\dot{\theta}\sin\theta)\mathbf{e}_y = \mathbf{0} \end{aligned} \tag{2.95}$$

Therefore,

$$^{\mathcal{F}}\mathbf{a}_{A/O} = -l(\ddot{\theta}\sin\theta + \dot{\theta}^2\cos\theta)\mathbf{e}_y \tag{2.96}$$

The velocity of point P relative to point A as viewed by an observer fixed to the ground is obtained by applying the transport theorem from reference frame $\mathcal{B}$ to reference frame $\mathcal{F}$ as

$$^{\mathcal{F}}\mathbf{v}_{P/A} = \frac{^{\mathcal{F}}d}{dt}(\mathbf{r}_{P/A}) = \frac{^{\mathcal{B}}d}{dt}(\mathbf{r}_{P/A}) + {}^{\mathcal{F}}\boldsymbol{\omega}^{\mathcal{B}} \times \mathbf{r}_{P/A} \tag{2.97}$$

Now

$$^{\mathcal{F}}\boldsymbol{\omega}^{\mathcal{B}} = {}^{\mathcal{F}}\boldsymbol{\omega}^{\mathcal{A}} + {}^{\mathcal{A}}\boldsymbol{\omega}^{\mathcal{B}} \tag{2.98}$$

where

$$^{\mathcal{A}}\boldsymbol{\omega}^{\mathcal{B}} = \dot{\theta}\mathbf{u}_z \tag{2.99}$$

Therefore,

$$\begin{aligned} ^{\mathcal{F}}\boldsymbol{\omega}^{\mathcal{B}} &= \Omega\mathbf{e}_y + \dot{\theta}\mathbf{u}_z = \Omega(-\cos\theta\,\mathbf{u}_r + \sin\theta\,\mathbf{u}_\theta) + \dot{\theta}\mathbf{u}_z \\ &= -\Omega\cos\theta\,\mathbf{u}_r + \Omega\sin\theta\,\mathbf{u}_\theta + \dot{\theta}\mathbf{u}_z \end{aligned} \tag{2.100}$$

Now we have

$$\begin{aligned} \frac{^{\mathcal{B}}d}{dt}(\mathbf{r}_{P/A}) &= \mathbf{0} \\ ^{\mathcal{F}}\boldsymbol{\omega}^{\mathcal{B}} \times \mathbf{r}_{P/A} &= (-\Omega\cos\theta\,\mathbf{u}_r + \Omega\sin\theta\,\mathbf{u}_\theta + \dot{\theta}\mathbf{u}_z) \times \tfrac{l}{2}\mathbf{u}_r \\ &= \frac{l\dot{\theta}}{2}\mathbf{u}_\theta - \frac{l\Omega\sin\theta}{2}\mathbf{u}_z \end{aligned} \tag{2.101}$$

Therefore,

$$^{\mathcal{F}}\mathbf{v}_{P/A} = \frac{l\dot{\theta}}{2}\mathbf{u}_\theta - \frac{l\Omega\sin\theta}{2}\mathbf{u}_z \tag{2.102}$$

The acceleration of point P relative to point A as viewed by an observer fixed to the ground is then given as

$$^{\mathcal{F}}\mathbf{a}_{P/A} = \frac{^{\mathcal{F}}d}{dt}\left(^{\mathcal{F}}\mathbf{v}_{P/A}\right) = \frac{^{\mathcal{B}}d}{dt}\left(^{\mathcal{F}}\mathbf{v}_{P/A}\right) + {}^{\mathcal{F}}\boldsymbol{\omega}^{\mathcal{B}} \times {}^{\mathcal{F}}\mathbf{v}_{P/A} \tag{2.103}$$

Now we have

$$\begin{aligned}
{}^{\mathcal{B}}\frac{d}{dt}\left({}^{\mathcal{F}}\mathbf{v}_{P/A}\right) &= \frac{l\ddot{\theta}}{2}\mathbf{u}_\theta - \frac{l\Omega\dot{\theta}\cos\theta}{2}\mathbf{u}_z \\
{}^{\mathcal{F}}\boldsymbol{\omega}^{\mathcal{B}} \times {}^{\mathcal{F}}\mathbf{v}_{P/A} &= (-\Omega\cos\theta\,\mathbf{u}_r + \Omega\sin\theta\,\mathbf{u}_\theta + \dot{\theta}\mathbf{u}_z) \times (\frac{l\dot{\theta}}{2}\mathbf{u}_\theta - \frac{l\Omega\sin\theta}{2}\mathbf{u}_z)
\end{aligned} \tag{2.104}$$

The second term in Eq. (2.104) can be simplified to

$${}^{\mathcal{F}}\boldsymbol{\omega}^{\mathcal{B}} \times {}^{\mathcal{F}}\mathbf{v}_{P/A} = -\frac{l\Omega\dot{\theta}\cos\theta}{2}\mathbf{u}_z - \frac{l\Omega^2\cos\theta\,\sin\theta}{2}\mathbf{u}_\theta - \left(\frac{l\Omega^2\sin^2\theta + l\dot{\theta}^2}{2}\right)\mathbf{u}_r \tag{2.105}$$

Adding the first term in Eq. (2.104) to the result of Eq. (2.105), we obtain the acceleration of point P relative to point A as viewed by an observer fixed to the ground as

$${}^{\mathcal{F}}\mathbf{a}_{P/A} = -\left(\frac{l\Omega^2\sin^2\theta + l\dot{\theta}^2}{2}\right)\mathbf{u}_r + \left(\frac{l\ddot{\theta}}{2} - \frac{l\Omega^2\cos\theta\,\sin\theta}{2}\right)\mathbf{u}_\theta - l\Omega\dot{\theta}\cos\theta\,\mathbf{u}_z \tag{2.106}$$

Using the aforementioned results, we obtain the velocity and acceleration of point P relative to point O as viewed by an observer fixed to the ground as follows. First, adding the results of Eqs. (2.93) and (2.102), we obtain

$${}^{\mathcal{F}}\mathbf{v}_{P/O} = -l\dot{\theta}\sin\theta\,\mathbf{e}_y + \frac{l\dot{\theta}}{2}\mathbf{u}_\theta - \frac{l\Omega\sin\theta}{2}\mathbf{u}_z \tag{2.107}$$

Finally, adding the results of Eqs. (2.96) and (2.106), we obtain

$$\begin{aligned}
{}^{\mathcal{F}}\mathbf{a}_{P/O} = {} & -l(\ddot{\theta}\sin\theta + \dot{\theta}^2\cos\theta)\mathbf{e}_y \\
& -\left(\frac{l\Omega^2\sin^2\theta + l\dot{\theta}^2}{2}\right)\mathbf{u}_r + \left(\frac{l\ddot{\theta}}{2} - \frac{l\Omega^2\cos\theta\,\sin\theta}{2}\right)\mathbf{u}_\theta \\
& - l\Omega\dot{\theta}\cos\theta\,\mathbf{u}_z
\end{aligned} \tag{2.108}$$

A last point pertains to the velocity of point B. It was stated in the problem that, "point B moves with constant velocity $\mathbf{v}_0$ relative to the horizontal shaft." Now, because the horizontal shaft is fixed in reference frame $\mathcal{A}$, we have

$${}^{\mathcal{A}}\mathbf{v}_B = v_0\mathbf{e}_x = \text{constant} \tag{2.109}$$

Another expression for the ${}^{\mathcal{A}}\mathbf{v}_B$ is obtained as follows. First,

$$\mathbf{r}_B = l\sin\theta\,\mathbf{e}_x \tag{2.110}$$

Therefore,

$${}^{\mathcal{A}}\mathbf{v}_B = l\dot{\theta}\cos\theta\,\mathbf{e}_x \tag{2.111}$$

Equating the expressions in Eq. (2.109) and (2.111), we obtain

$$v_0 = l\dot{\theta}\cos\theta \tag{2.112}$$

which implies that

$$\dot{\theta} = \frac{v_0}{l\cos\theta} = \frac{v_0}{l}\sec\theta \tag{2.113}$$

Differentiating Eq. (2.113) with respect to time, we have

$$\ddot{\theta} = \frac{v_0}{l}\dot{\theta}\sec\theta\tan\theta = \frac{v_0^2}{l^2}\sec^2\theta\tan\theta \tag{2.114}$$

The expressions for $\dot{\theta}$ and $\ddot{\theta}$ given in Eqs. (2.113) and (2.114), respectively, can be substituted into the expressions for ${}^{\mathcal{F}}\mathbf{v}_{P/O}$ and ${}^{\mathcal{F}}\mathbf{a}_{P/O}$ to obtain expressions that do not involve either $\dot{\theta}$ or $\ddot{\theta}$.

Question 2–10

A circular disk of radius R is attached to a rotating shaft of length L as shown in Fig. P2-10. The shaft rotates about the horizontal direction with a constant angular velocity $\mathbf{\Omega}$ relative to the ground. The disk, in turn, rotates about its center about an axis orthogonal to the shaft. Knowing that the angle θ describes the position of a point P located on the edge of the disk relative to the center of the disk, determine the velocity and acceleration of point P relative to the ground.

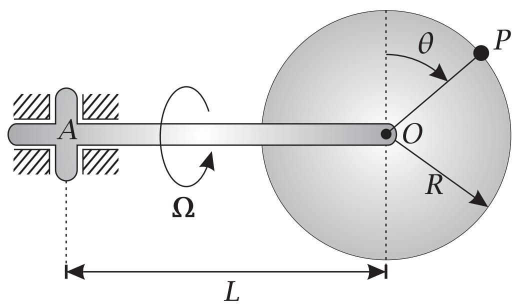

Figure P2-10

Solution to Question 2–10

First, let $\mathcal{F}$ be a reference frame fixed to the ground. Then, we choose the following coordinate system fixed in reference frame $\mathcal{F}$:

Origin at Point O		
$\mathbf{E}_2$	=	Along AO
$\mathbf{E}_3$	=	Orthogonal to Disk and Into Page at $t = 0$
$\mathbf{E}_1$	=	$\mathbf{E}_2 \times \mathbf{E}_3$

Next, let $\mathcal{A}$ be a reference frame fixed to the horizontal shaft. Then, we choose the following coordinate system fixed in reference frame $\mathcal{F}$:

Origin at Point O		
$\mathbf{e}_2$	=	Along AO
$\mathbf{e}_3$	=	Orthogonal to Disk and Into Page
$\mathbf{e}_1$	=	$\mathbf{e}_2 \times \mathbf{e}_3$

Lastly, let $\mathcal{B}$ be a reference frame fixed to the disk. Then, choose the following coordinate system fixed in reference frame $\mathcal{B}$:

$$
\begin{array}{lcl}
 & \text{Origin at Point } O & \\
\mathbf{u}_1 & = & \text{Along } OP \\
\mathbf{u}_3 & = & \text{Orthogonal to Disk} \\
 & & \text{and Into Page} \\
\mathbf{u}_2 & = & \mathbf{u}_3 \times \mathbf{u}_1
\end{array}
$$

Now, since the shaft rotates with angular velocity $\mathbf{\Omega}$ relative to the ground, the angular velocity of reference frame $\mathcal{A}$ in reference frame $\mathcal{F}$ is given as

$$ {}^{\mathcal{F}}\boldsymbol{\omega}^{\mathcal{A}} = \mathbf{\Omega} = \Omega \mathbf{e}_2 \tag{2.115} $$

Furthermore, since the disk rotates with angular rate $\dot{\theta}$ relative to the shaft, the angular velocity of reference frame $\mathcal{B}$ in reference frame $\mathcal{A}$ is given as

$$ {}^{\mathcal{A}}\boldsymbol{\omega}^{\mathcal{B}} = \dot{\theta}\mathbf{u}_3 \tag{2.116} $$

Finally, the geometry of the bases $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ and $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ is shown in Fig. (2.117). Using Fig. (2.117), we have that

$$
\begin{array}{lcl}
\mathbf{e}_1 & = & \cos\theta\mathbf{u}_1 - \sin\theta\mathbf{u}_2 \\
\mathbf{e}_2 & = & \sin\theta\mathbf{u}_1 + \cos\theta\mathbf{u}_2
\end{array} \tag{2.117}
$$

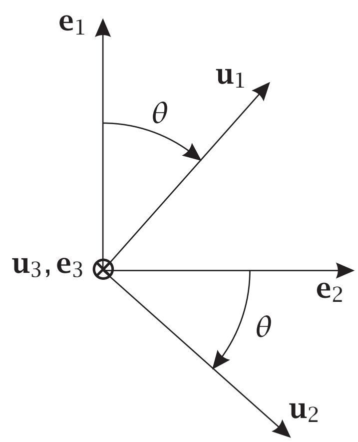

Figure 2-2 Relationship Between Basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ and $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ for Question 2–10

Given the definitions of the reference frames and coordinate systems, the position of point P is given as

$$ \mathbf{r} = R\mathbf{u}_1 \tag{2.118} $$

The velocity of point P in reference frame $\mathcal{F}$ is then given as

$$ {}^{\mathcal{F}}\mathbf{v} = \frac{{}^{\mathcal{F}}d\mathbf{r}}{dt} = \frac{{}^{\mathcal{F}}d}{dt}(R\mathbf{u}_1) \tag{2.119}$$

Now, since the basis $\{\mathbf{u}_1,\mathbf{u}_2,\mathbf{u}_3\}$ is fixed in reference frame $\mathcal{F}$, it is convenient to apply the rate of change transport theorem of Eq. (2–128) between reference frame $\mathcal{B}$ and reference frame $\mathcal{F}$ as

$$ \frac{{}^{\mathcal{F}}d}{dt}(R\mathbf{u}_1) = \frac{{}^{\mathcal{B}}d}{dt}(R\mathbf{u}_1) + {}^{\mathcal{F}}\boldsymbol{\omega}^{\mathcal{B}} \times R\mathbf{u}_1 \tag{2.120}$$

First, since R is constant and the basis $\{\mathbf{u}_1,\mathbf{u}_2,\mathbf{u}_3\}$ is fixed in reference frame $\mathcal{B}$, we have that

$$ \frac{{}^{\mathcal{B}}d}{dt}(R\mathbf{u}_1) = \mathbf{0} \tag{2.121}$$

Next, applying the angular velocity addition rule of Eq. (2–136), we obtain ${}^{\mathcal{F}}\boldsymbol{\omega}^{\mathcal{B}}$ as

$$ {}^{\mathcal{F}}\boldsymbol{\omega}^{\mathcal{B}} = {}^{\mathcal{F}}\boldsymbol{\omega}^{\mathcal{A}} + {}^{\mathcal{A}}\boldsymbol{\omega}^{\mathcal{B}} = \Omega\mathbf{e}_2 + \dot{\theta}\mathbf{u}_3 \tag{2.122}$$

Using ${}^{\mathcal{F}}\boldsymbol{\omega}^{\mathcal{B}}$ from Eq. (2.122), we obtain ${}^{\mathcal{F}}\boldsymbol{\omega}^{\mathcal{B}} \times R\mathbf{u}_1$ as

$$ {}^{\mathcal{F}}\boldsymbol{\omega}^{\mathcal{B}} \times R\mathbf{u}_1 = (\Omega\mathbf{e}_2 + \dot{\theta}\mathbf{u}_3) \times R\mathbf{u}_1 = R\Omega\mathbf{e}_2 \times \mathbf{u}_1 + R\dot{\theta}\mathbf{u}_2 \tag{2.123}$$

Then, from Eq. (2.117), we have that

$$ \mathbf{e}_2 \times \mathbf{u}_1 = (\sin\theta\,\mathbf{u}_1 + \cos\theta\,\mathbf{u}_2) \times \mathbf{u}_1 = -\cos\theta\,\mathbf{u}_3 \tag{2.124}$$

Substituting the result of Eq. (2.124) into Eq. (2.123), we obtain

$$ {}^{\mathcal{F}}\boldsymbol{\omega}^{\mathcal{B}} \times R\mathbf{u}_1 = -R\Omega\cos\theta\,\mathbf{u}_3 + R\dot{\theta}\mathbf{u}_2 \tag{2.125}$$

Adding Eq. (2.121) and Eq. (2.125), we obtain the velocity of point P in reference frame $\mathcal{F}$ as

$$ {}^{\mathcal{F}}\mathbf{v} = R\dot{\theta}\mathbf{u}_2 - R\Omega\cos\theta\,\mathbf{u}_3 \tag{2.126}$$

Next, the acceleration of point P in reference frame $\mathcal{F}$ is given as

$$ {}^{\mathcal{F}}\mathbf{a} = \frac{{}^{\mathcal{F}}d}{dt}\left({}^{\mathcal{F}}\mathbf{v}\right) \tag{2.127}$$

It is seen that the expression for ${}^{\mathcal{F}}\mathbf{v}$ is given in terms of the basis $\{\mathbf{u}_1,\mathbf{u}_2,\mathbf{u}_3\}$ where $\{\mathbf{u}_1,\mathbf{u}_2,\mathbf{u}_3\}$ is fixed in reference frame $\mathcal{B}$. Thus, applying the rate of change transport theorem of Eq. (2–128) between reference frame $\mathcal{B}$ and $\mathcal{F}$ to ${}^{\mathcal{F}}\mathbf{v}$, we obtain

$$ {}^{\mathcal{F}}\mathbf{a} = \frac{{}^{\mathcal{F}}d}{dt}\left({}^{\mathcal{F}}\mathbf{v}\right) = \frac{{}^{\mathcal{B}}d}{dt}\left({}^{\mathcal{F}}\mathbf{v}\right) + {}^{\mathcal{F}}\boldsymbol{\omega}^{\mathcal{B}} \times {}^{\mathcal{F}}\mathbf{v} \tag{2.128}$$

Now, observing that R and Ω are constant, the first term in Eq. (2.128) is given as

$$\frac{{}^{\mathcal{B}}d}{dt}\left({}^{\mathcal{F}}\mathbf{v}\right) = R\ddot{\theta}\mathbf{u}_2 + R\Omega\dot{\theta}\sin\theta\,\mathbf{u}_3 \tag{2.129}$$

Next, using ${}^{\mathcal{F}}\boldsymbol{\omega}^{\mathcal{B}}$ from Eq. (2.122), we obtain the second term in Eq. (2.128) as

$${}^{\mathcal{F}}\boldsymbol{\omega}^{\mathcal{B}} \times {}^{\mathcal{F}}\mathbf{v} = (\Omega\mathbf{e}_2 + \dot{\theta}\mathbf{u}_3) \times (-R\Omega\cos\theta\,\mathbf{u}_3 + R\dot{\theta}\mathbf{u}_2) \tag{2.130}$$

Expanding Eq. (2.130), we obtain

$${}^{\mathcal{F}}\boldsymbol{\omega}^{\mathcal{B}} \times {}^{\mathcal{F}}\mathbf{v} = R\Omega\dot{\theta}\mathbf{e}_2 \times \mathbf{u}_2 - R\Omega^2\cos\theta\,\mathbf{e}_2 \times \mathbf{u}_3 - R\dot{\theta}^2\mathbf{u}_1 \tag{2.131}$$

Then, using the expression for $\mathbf{e}_2$ from Eq. (2.117), we obtain

$$\begin{array}{rcl} \mathbf{e}_2 \times \mathbf{u}_2 & = & (\sin\theta\,\mathbf{u}_1 + \cos\theta\,\mathbf{u}_2) \times \mathbf{u}_2 = \sin\theta\,\mathbf{u}_3 \\ \mathbf{e}_2 \times \mathbf{u}_3 & = & (\sin\theta\,\mathbf{u}_1 + \cos\theta\,\mathbf{u}_2) \times \mathbf{u}_3 = \cos\theta\,\mathbf{u}_1 - \sin\theta\,\mathbf{u}_2 \end{array} \tag{2.132}$$

Substituting the results of Eq. (2.132) into Eq. (2.131), we obtain

$${}^{\mathcal{F}}\boldsymbol{\omega}^{\mathcal{B}} \times {}^{\mathcal{F}}\mathbf{v} = R\Omega\dot{\theta}\sin\theta\,\mathbf{u}_3 - R\Omega^2\cos\theta\,(\cos\theta\,\mathbf{u}_1 - \sin\theta\,\mathbf{u}_2) - R\dot{\theta}^2\mathbf{u}_1 \tag{2.133}$$

Adding the expressions in Eq. (2.129) and Eq. (2.133), we obtain the acceleration of point P in reference frame $\mathcal{F}$ as

$${}^{\mathcal{F}}\mathbf{a} = R\ddot{\theta}\mathbf{u}_2 + R\Omega\dot{\theta}\sin\theta\,\mathbf{u}_3 + R\Omega\dot{\theta}\sin\theta\,\mathbf{u}_3 - R\Omega^2\cos\theta\,(\cos\theta\,\mathbf{u}_1 - \sin\theta\,\mathbf{u}_2) - R\dot{\theta}^2\mathbf{u}_1 \tag{2.134}$$

Simplifying Eq. (2.134), we obtain

$${}^{\mathcal{F}}\mathbf{a} = -(R\Omega^2\cos^2\theta + R\dot{\theta}^2)\mathbf{u}_1 + (R\ddot{\theta} + R\Omega^2\cos\theta\,\sin\theta\,)\mathbf{u}_2 + 2R\Omega\dot{\theta}\sin\theta\,\mathbf{u}_3 \tag{2.135}$$