Chapter 2

Kinematics

Question 2–1

A bug B crawls radially outward at constant speed \( v_0 \) from the center of a rotating disk as shown in Fig. P2-1. Knowing that the disk rotates about its center \( O \) with constant absolute angular velocity \( \Omega \) relative to the ground (where \( ||\Omega|| = \Omega \)), determine the velocity and acceleration of the bug as viewed by an observer fixed to the ground.

Figure P2-1

Solution to Question 2–1

For this problem it is convenient to choose a fixed reference frame \( F \) and a non-inertial reference frame \( A \) that is fixed in the disk. Corresponding to reference frame \( F \) we choose the following coordinate system:

<table>
<thead>
<tr>
<th>Origin at Point ( O )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( E_x ) ( = ) \hline</td>
</tr>
<tr>
<td>( E_z ) ( = ) \hline</td>
</tr>
<tr>
<td>( E_y ) ( = ) \hline</td>
</tr>
</tbody>
</table>

Along \( OB \) at Time \( t = 0 \)
Out of Page
\( E_z \times E_x \)
Corresponding to the reference frame $\mathcal{A}$ that is fixed in the disk, we choose the following coordinate system

- Origin at Point $O$
- $e_x = \text{Along } OB$
- $e_z = \text{Out of Page } (= E_z)$
- $e_y = e_z \times e_x$

The position of the bug is then resolved in the basis $\{e_x, e_y, e_z\}$ as

$$\mathbf{r} = r e_x$$  \hspace{1cm} (2.1)

Now, since the platform rotates about the $e_z$-direction relative to the ground, the angular velocity of reference frame $\mathcal{A}$ in reference frame $\mathcal{F}$ is given as

$$\mathcal{F} \omega^A = \Omega e_z$$  \hspace{1cm} (2.2)

The velocity is found by applying the basic kinematic equation. This gives

$$\mathcal{F} \mathbf{v} = \frac{d}{dt} \mathcal{F} \mathbf{r} = \dot{\mathcal{A}} \mathbf{r} + \mathcal{F} \omega^A \times \mathbf{r}$$  \hspace{1cm} (2.3)

Now we have

$$\dot{\mathcal{A}} \mathbf{r} = \dot{r} e_x = v_0 e_x$$  \hspace{1cm} (2.4)

$$\mathcal{F} \omega^A \times \mathbf{r} = \Omega e_z \times r e_x = r \Omega e_y$$  \hspace{1cm} (2.5)

Adding Eqs. (2.4) and (2.5), we obtain the velocity of the bug in reference frame $\mathcal{F}$ as

$$\mathcal{F} \mathbf{v} = v_0 e_x + r \Omega e_y$$  \hspace{1cm} (2.6)

The acceleration is found by applying the basic kinematic equation to $\mathcal{F} \mathbf{v}$. This gives

$$\mathcal{F} \mathbf{a} = \frac{d}{dt} \mathcal{F} \mathbf{v} = \dot{\mathcal{A}} \mathcal{F} \mathbf{v} + \mathcal{F} \omega^A \times \mathcal{F} \mathbf{v}$$  \hspace{1cm} (2.7)

Using $\mathcal{F} \mathbf{v}$ from Eq. (2.6) and noting that $v_0$ and $\Omega$ are constant, we have that

$$\dot{\mathcal{A}} \mathcal{F} \mathbf{v} = \dot{\Omega} e_y = \Omega v_0 e_y$$

$$\mathcal{F} \omega^A \times \mathcal{F} \mathbf{v} = \Omega e_z \times [v_0 e_x + r \Omega e_y]$$

$$= -\Omega^2 r e_x + \Omega v_0 e_y$$  \hspace{1cm} (2.8)

Therefore, the acceleration in reference frame $\mathcal{F}$ is given as

$$\mathcal{F} \mathbf{a} = -\Omega^2 r e_x + 2\Omega v_0 e_y$$  \hspace{1cm} (2.9)
**Question 2–2**

A particle, denoted by \( P \), slides on a circular table as shown in Fig. P2-2. The position of the particle is known in terms of the radius \( r \) measured from the center of the table at point \( O \) and the angle \( \theta \) where \( \theta \) is measured relative to the direction of \( OQ \) where \( Q \) is a point on the circumference of the table. Knowing that the table rotates with constant angular rate \( \Omega \), determine the velocity and acceleration of the particle as viewed by an observer in a fixed reference frame.

![Figure P2-2](image)

**Solution to Question 2–2**

For this problem it is convenient to define a fixed inertial reference frame \( F \) and two non-inertial reference frames \( A \) and \( B \). The first non-inertial reference frame \( A \) is fixed to the disk while the second non-inertial reference frame \( B \) is fixed to the direction of \( OP \). Corresponding to the fixed inertial reference frame \( F \), we choose the following coordinate system:

\[
\begin{align*}
\text{Origin at point } O & \\
E_x &= \text{Along } Ox \text{ at } t = 0 \\
E_z &= \text{Out of Page} \\
E_y &= E_z \times E_x
\end{align*}
\]

Corresponding to non-inertial reference frame \( A \), we choose the following coordinate system:

\[
\begin{align*}
\text{Origin at point } O & \\
e_x &= \text{Along } OQ \\
e_z &= \text{Out of Page } (= E_z) \\
e_y &= e_z \times e_x
\end{align*}
\]
Finally, corresponding to reference frame $B$, we choose the following coordinate system:

- **Origin at point $O$**
- $e_r = \text{Along } OP$
- $e_z = \text{Out of Page}$
- $e_\theta = e_z \times e_r$

Then, the position of the particle can be described in terms of the basis $\{e_r, e_\theta, e_z\}$ as

$$ r = re_r. \quad (2.10) $$

Now, in order to compute the velocity of the particle, it is necessary to apply the basic kinematic equation. In this case since we are interested in motion as viewed by an observer in the fixed inertial reference frame $F$, we need to determine the angular velocity of $B$ in $F$. First, since $A$ rotates relative to $F$ with angular velocity $\Omega$, we have that

$$ F\omega^A = \Omega = \Omega e_z \quad (2.11) $$

Next, since $B$ rotates relative to $A$ with angular rate $\dot{\theta}$ about the $e_z$-direction, we have that

$$ A\omega^B = \dot{\theta}e_z \quad (2.12) $$

Then, applying the theorem of addition of angular velocities, we have that

$$ F\omega^B = F\omega^A + A\omega^B = \Omega e_z + \dot{\theta}e_z = (\Omega + \dot{\theta})e_z \quad (2.13) $$

The velocity in reference frame is then found by applying the rate of change transport theorem as

$$ F\vec{v} = \frac{F\vec{d}r}{dt} = \frac{B\vec{d}r}{dt} + F\omega^B \times r \quad (2.14) $$

Now we have

$$ \frac{B\vec{d}r}{dt} = \dot{r}e_r \quad (2.15) $$

$$ F\omega^B \times r = (\Omega + \dot{\theta})e_z \times re_r = r(\Omega + \dot{\theta})e_\theta \quad (2.16) $$

Adding Eqs. (2.15) and (2.16), we obtain the velocity of the particle in reference frame $F$ as

$$ F\vec{v} = \dot{r}e_r + r(\Omega + \dot{\theta})e_\theta \quad (2.17) $$

The acceleration is found by applying the rate of change transport theorem to $F\vec{v}$. This gives

$$ F\vec{a} = \frac{F\vec{d}r}{dt} (F\vec{v}) = \frac{B\vec{d}r}{dt} (F\vec{v}) + F\omega^B \times F\vec{v} \quad (2.18) $$
Using $\mathcal{F}v$ from Eq. (2.17) and noting again that $\Omega$ is constant, we have

\[
\frac{d}{dt}(\mathcal{F}v) = \dot{r} \mathbf{e}_r + \left[ \dot{\rho} (\Omega + \dot{\theta}) + r \ddot{\theta} \right] \mathbf{e}_\theta
\]  
(2.19)

\[
\mathcal{F} \omega^B \times \mathcal{F}v = (\Omega + \dot{\theta}) \mathbf{e}_z \times \left[ \dot{r} \mathbf{e}_r + r (\Omega + \dot{\theta}) \mathbf{e}_\theta \right] = -r (\Omega + \dot{\theta})^2 \mathbf{e}_r + \dot{r} (\Omega + \dot{\theta}) \mathbf{e}_\theta
\]  
(2.20)

Adding Eqs. (2.19) and (2.20), we obtain the acceleration of the particle in reference frame $\mathcal{F}$ as

\[
\mathcal{F}a = \left[ \ddot{r} - r (\Omega + \dot{\theta})^2 \right] \mathbf{e}_r + \left[ r \ddot{\theta} + 2 \dot{r} (\Omega + \dot{\theta}) \right] \mathbf{e}_\theta
\]  
(2.21)
Question 2–3

A collar slides along a rod as shown in Fig. P2-3. The rod is free to rotate about a hinge at the fixed point $O$. Simultaneously, the rod rotates about the vertical direction with constant angular velocity $\Omega$ relative to the ground. Knowing that $r$ describes the location of the collar along the rod, that $\theta$ is the angle measured from the vertical, and that $\Omega = \|\Omega\|$, determine the velocity and acceleration of the collar as viewed by an observer fixed to the ground.

![Figure P2-3](image)

Solution to Question 2–3

First, let $\mathcal{F}$ be a fixed reference frame. Then, choose the following coordinate system fixed in reference frame $\mathcal{F}$:

Origin at point $O$
\[
E_x = \text{Along } \Omega \\
E_z = \text{Orthogonal to Plane of} \\
\text{Shaft and Arm at } t = 0 \\
E_y = E_z \times E_x
\]

Next, let $\mathcal{A}$ be a reference frame fixed to the vertical shaft. Then, choose the following coordinate system fixed in reference frame $\mathcal{A}$:

Origin at point $O$
\[
e_x = \text{Along } \Omega \\
e_z = \text{Orthogonal to Plane of} \\
\text{Shaft and Arm} \\
e_y = e_z \times e_x
\]
Finally, let $\mathcal{B}$ be a reference frame fixed to the rod. Then, choose the following coordinate system fixed in reference frame $\mathcal{B}$:

- **Origin at point $O$**
  - $e_r = \text{Along } OP$
  - $e_z = u_z$
  - $e_\theta = e_z \times e_r$

The geometry of the bases $\{e_x, e_y, e_z\}$ and $\{e_r, e_\theta, e_z\}$ is shown in Fig. 2-1. Using Fig. 2-1, the relationship between the basis $\{e_x, e_y, e_z\}$ and $\{e_r, e_\theta, e_z\}$ is given as

\[
\begin{align*}
    e_x &= \cos \theta e_r - \sin \theta e_\theta \\
    e_y &= \sin \theta e_r + \cos \theta e_\theta
\end{align*}
\]  

(2.22)

![Figure 2-1](geom.png)  

**Figure 2-1** Geometry of Bases $\{e_x, e_y, e_z\}$ and $\{e_r, e_\theta, e_z\}$ for Question 2–3.

The position of the particle can then be expressed in the basis $\{e_r, e_\theta, e_z\}$ as

\[
r = re_r
\]

(2.23)

Now, since $\{e_r, e_\theta, e_z\}$ is fixed in reference frame $\mathcal{B}$, and we are interested in obtaining the velocity and acceleration as viewed by an observer fixed in the ground (i.e., reference frame $\mathcal{F}$), we need to obtain an expression for the angular velocity of reference frame $\mathcal{B}$ in reference frame $\mathcal{F}$. First, since reference frame $\mathcal{A}$ rotates relative to reference frame $\mathcal{B}$ with angular velocity $\Omega$ and $\Omega$ lies along the $e_x$-direction, we have that

\[
\mathcal{F} \omega^\mathcal{A} = \Omega = \Omega e_x
\]

(2.24)

Next, since reference frame $\mathcal{B}$ rotates relative to reference frame $\mathcal{A}$ with angular rate $\dot{\theta}$ about the $e_z$-direction. Therefore,

\[
\mathcal{A} \omega^\mathcal{B} = \dot{\theta} e_z
\]

(2.25)
Then, using the angular velocity addition theorem, we have the angular velocity of reference frame $B$ in reference frame $F$ as

$$F\omega^B = F\omega^A + A\omega^B = \Omega e_x + \dot{\theta} e_z$$

(2.26)

Now, since we have determined that the position of the collar is expressed most conveniently in terms of the basis $\{e_r, e_\theta, e_z\}$, it is also most convenient to express $F\omega^B$ in terms of the basis $\{e_r, e_\theta, e_z\}$. In particular, substituting the expression for $e_x$ from Eq. (2.22) into Eq. (2.26), we obtain $F\omega^B$ as

$$F\omega^B = \Omega (\cos \theta e_r - \sin \theta e_\theta) + \dot{\theta} e_z = \Omega \cos \theta e_r - \Omega \sin \theta e_\theta + \dot{\theta} e_z$$

(2.27)

The velocity in reference frame $F$ is then found by applying the rate of change transport theorem between reference frames $B$ and $F$ as

$$Fv = \frac{Fdr}{dt} = \frac{Bdr}{dt} + F\omega^B \times r$$

(2.28)

Now we have that

$$\frac{Bdr}{dt} = \dot{r} e_r$$

(2.29)

$$F\omega^B \times r = (\Omega \cos \theta e_r - \Omega \sin \theta e_\theta + \dot{\theta} e_z) \times re_r = r \Omega \sin \theta e_z + r \dot{\theta} e_\theta$$

(2.30)

Adding Eq. (2.29) and Eq. (2.30), we obtain the velocity of the collar in reference frame $F$ as

$$Fv = \dot{r} e_r + r \dot{\theta} e_\theta + r \Omega \sin \theta e_z$$

(2.31)

The acceleration of the collar is then obtained by applying the rate of change transport theorem to $Fv$ between reference frames $B$ and $F$ as

$$Fa = \frac{Fd}{dt} (Fv) = \frac{Bd}{dt} (Fv) + F\omega^B \times Fv$$

(2.32)

Now we have

$$\frac{Bd}{dt} (Fv) = \dot{r} e_r + (\dot{\theta} + r \ddot{\theta}) e_\theta + \left[\Omega (\dot{r} \sin \theta + r \ddot{\theta} \cos \theta)\right] e_z$$

(2.33)

$$F\omega^B \times Fv = (\Omega \cos \theta e_r - \Omega \sin \theta e_\theta + \dot{\theta} e_z) \times (\dot{r} e_r + r \dot{\theta} e_\theta + r \Omega \sin \theta e_z) = r \Omega \dot{\theta} \cos \theta e_z - r \Omega^2 \cos \theta \sin \theta e_\theta + \dot{\theta} e_\theta + \Omega \dot{\theta} \sin \theta e_z - r \Omega^2 \sin^2 \theta e_r$$

$$+ r \Omega \dot{\theta} e_\theta - \dot{\theta} e_z$$

$$= - (r \ddot{\theta} + r \Omega^2 \sin^2 \theta) e_r + (\dot{\theta} r - r \Omega^2 \cos \theta \sin \theta) e_\theta$$

$$+ (r \Omega \dot{\theta} \cos \theta + \dot{\theta} \Omega \sin \theta) e_z$$

(2.34)

Adding Eqs. (2.33) and (2.34), we obtain the acceleration of the collar in reference frame $F$ as

$$Fa = (\ddot{r} - r \dddot{\theta} - r \Omega^2 \sin^2 \theta) e_r + (2 \dot{\theta} + r \dddot{\theta} - r \Omega^2 \cos \theta \sin \theta) e_\theta$$

$$+ 2 \Omega (\dot{r} \sin \theta + r \dot{\theta} \cos \theta) e_z$$

(2.35)
**Question 2–4**

A particle slides along a track in the form of a parabola $y = x^2/a$ as shown in Fig. P2-4. The parabola rotates about the vertical with a constant angular velocity $\Omega$ relative to a fixed reference frame (where $\Omega = \|\Omega\|$). Determine the velocity and acceleration of the particle as viewed by an observer in a fixed reference frame.

![Figure P2-4](image)

**Solution to Question 2–4**

For this problem it is convenient to define a fixed inertial reference frame $\mathcal{F}$ and a non-inertial reference frame $\mathcal{A}$. Corresponding to reference frame $\mathcal{F}$, we choose the following coordinate system:

- Origin at Point $O$
  - $E_x = \text{Along } OQ \text{ When } t = 0$
  - $E_y = \text{Along } Oy \text{ When } t = 0$
  - $E_z = E_x \times E_y$

Furthermore, corresponding to reference frame $\mathcal{A}$, we choose the following coordinate system:

- Origin at Point $O$
  - $e_x = \text{Along } OQ$
  - $e_y = \text{Along } Oy$
  - $e_z = e_x \times e_y$

The position of the particle is then given in terms of the basis $\{e_x, e_y, e_z\}$ as

$$r = xe_x + ye_y = xe_x + (x^2/a)e_y \quad (2.36)$$

Furthermore, since the parabola spins about the $e_y$-direction, the angular velocity of reference frame $\mathcal{A}$ in reference frame $\mathcal{F}$ is given as

$$\mathcal{F}\omega^\mathcal{A} = \Omega = \Omega e_y \quad (2.37)$$
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The velocity in reference frame $\mathcal{F}$ is then found using the rate of change transport theorem as

$$\mathcal{F}\mathbf{v} = \frac{\mathcal{F}d\mathbf{r}}{dt} = \frac{\mathcal{A}d\mathbf{r}}{dt} + \mathcal{F}\mathbf{w}^A \times \mathbf{r} \quad (2.38)$$

Using $\mathbf{r}$ from Eq. (2.36) and $\mathcal{F}\mathbf{w}^A$ from Eq. (2.37), we have

$$\frac{\mathcal{A}d\mathbf{r}}{dt} = \mathbf{\dot{x}e}_x + (2\mathbf{x}\mathbf{\ddot{x}}/a)e_y \quad (2.39)$$

$$\mathcal{F}\mathbf{w}^A \times \mathbf{r} = \Omega\mathbf{e}_y \times (\mathbf{x}\mathbf{e}_x + (\mathbf{x}^2/a)e_y) = -\Omega \mathbf{x}\mathbf{e}_z \quad (2.40)$$

Adding Eqs. (2.39) and (2.40), we obtain $\mathcal{F}\mathbf{v}$ as

$$\mathcal{F}\mathbf{v} = \mathbf{\dot{x}e}_x + (2\mathbf{x}\mathbf{\ddot{x}}/a)e_y - \Omega \mathbf{x}\mathbf{e}_z \quad (2.41)$$

The acceleration in reference frame $\mathcal{F}$ is found by applying the rate of change transport theorem to $\mathcal{F}\mathbf{v}$ as

$$\mathcal{F}\mathbf{a} = \frac{\mathcal{F}d}{dt}(\mathcal{F}\mathbf{v}) = \frac{\mathcal{A}d}{dt}(\mathcal{F}\mathbf{v}) + \mathcal{F}\mathbf{w}^A \times \mathcal{F}\mathbf{v} \quad (2.42)$$

Using $\mathcal{F}\mathbf{v}$ from Eq. (2.41) and $\mathcal{F}\mathbf{w}^A$ from Eq. (2.37), we have

$$\frac{\mathcal{A}d}{dt}(\mathcal{F}\mathbf{v}) = \mathbf{\ddot{x}e}_x + \left[2(\mathbf{x}^2 + \mathbf{x}\mathbf{\ddot{x}})/a\right]e_y - \Omega \mathbf{x}\mathbf{e}_z \quad (2.43)$$

$$\mathcal{F}\mathbf{w}^A \times \mathcal{F}\mathbf{v} = \Omega\mathbf{e}_y \times (\mathbf{x}\mathbf{e}_x + (2\mathbf{x}\mathbf{\ddot{x}}/a)e_y - \Omega \mathbf{x}\mathbf{e}_z) = -\Omega \mathbf{x}\mathbf{e}_z - \Omega^2 \mathbf{x}\mathbf{e}_z \quad (2.44)$$

Adding Eq. (2.43) and (2.44), we obtain $\mathcal{F}\mathbf{a}$ as

$$\mathcal{F}\mathbf{a} = (\mathbf{\dddot{x}} - \Omega^2 \mathbf{x})\mathbf{e}_x + \left[2(\mathbf{\dddot{x}}^2 + \mathbf{x}\mathbf{\dddot{x}})/a\right] \mathbf{e}_y - 2\Omega \mathbf{x}\mathbf{e}_z \quad (2.45)$$
Question 2–5

A satellite is in motion over the Earth as shown in Fig. P2-5. The Earth is modeled as a sphere of radius $R$ that rotates with constant angular velocity $\Omega$ in a direction $e_z$ where $e_z$ lies along a radial line that lies in the direction from the center of the Earth at point $O$ to the North Pole of the Earth at point $N$. Furthermore, the center of the Earth is assumed to be an absolutely fixed point. The position of the satellite is known in terms of an Earth-centered Earth-fixed Cartesian coordinate system whose right-handed basis $\{e_x, e_y, e_z\}$ is defined as follows:

- The direction $e_x$ lies orthogonal to $e_z$ in the equatorial plane of the Earth along the line from $O$ to $P$ where $P$ lies at the intersection of the equator with the great circle called the Prime Meridian

- The direction $e_y$ lies orthogonal to both $e_x$ and $e_z$ in the equatorial plane of the Earth such that $e_y = e_z \times e_x$

Using the basis $\{e_x, e_y, e_z\}$ to express all quantities, determine the velocity and acceleration of the spacecraft (a) as viewed by an observer fixed to the Earth and (b) as viewed by an observer in a fixed inertial reference frame.

![Figure P2-5](image)

Solution to Question 2–5

First, let $\mathcal{F}$ be a fixed inertial reference frame. Next, let $\mathcal{A}$ be a reference frame that is fixed in the planet. Corresponding to reference frame $\mathcal{A}$, we choose the
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following coordinate system:

\[
\begin{align*}
\text{Origin at point } O \\
\mathbf{e}_x &= \text{Along } OP \\
\mathbf{e}_z &= \text{Along } ON \\
\mathbf{e}_y &= \mathbf{e}_z \times \mathbf{e}_x \quad (= \text{Along } OQ)
\end{align*}
\]

The position of the spacecraft is then given in terms of the basis \{\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z\} as

\[
r = x\mathbf{e}_x + y\mathbf{e}_y + z\mathbf{e}_z \quad (2.46)
\]

Now, since the planet rotates with constant angular velocity \(\Omega\) about the \(ON\)-direction relative to reference frame \(\mathcal{F}\), we have that

\[
\mathcal{F}\mathbf{\omega}^A = \Omega\mathbf{e}_z \quad (2.47)
\]

The velocity of the spacecraft is then found by applying the rate of change transport theorem as

\[
\mathcal{F}\mathbf{v} = \frac{\mathcal{A}\mathbf{d}}{\mathcal{d}t} (\mathcal{F}\mathbf{r}) + \mathcal{F}\mathbf{\omega}^A \times \mathbf{r} \quad (2.48)
\]

Now we have

\[
\frac{\mathcal{A}\mathbf{d}}{\mathcal{d}t} (\mathcal{F}\mathbf{r}) = \dot{x}\mathbf{e}_x + \dot{y}\mathbf{e}_y + \dot{z}\mathbf{e}_z \quad (2.49)
\]

\[
\mathcal{F}\mathbf{\omega}^A \times \mathbf{r} = \Omega\mathbf{e}_z \times (x\mathbf{e}_x + y\mathbf{e}_y + z\mathbf{e}_z) \]

\[
= \Omega x\mathbf{e}_y - \Omega y\mathbf{e}_x \quad (2.50)
\]

Adding Eqs. (2.49) and (2.50), we obtain \(\mathcal{F}\mathbf{v}\) as

\[
\mathcal{F}\mathbf{v} = (\dot{x} - \Omega\dot{y})\mathbf{e}_x + (\dot{y} + \Omega\dot{x})\mathbf{e}_y + \dot{z}\mathbf{e}_z \quad (2.51)
\]

Next, the acceleration of the spacecraft in reference frame \(\mathcal{F}\) is found by applying the rate of change transport theorem to \(\mathcal{F}\mathbf{v}\) as

\[
\mathcal{F}\mathbf{a} = \frac{\mathcal{A}\mathbf{d}}{\mathcal{d}t} (\mathcal{F}\mathbf{v}) = \frac{\mathcal{A}\mathbf{d}}{\mathcal{d}t} (\mathcal{F}\mathbf{v}) + \mathcal{F}\mathbf{\omega}^A \times \mathcal{F}\mathbf{v} \quad (2.52)
\]

Now we have

\[
\frac{\mathcal{A}\mathbf{d}}{\mathcal{d}t} (\mathcal{F}\mathbf{v}) = (\ddot{x} - \Omega\dot{y})\mathbf{e}_x + (\ddot{y} + \Omega\dot{x})\mathbf{e}_y + \ddot{z}\mathbf{e}_z \quad (2.53)
\]

\[
\mathcal{F}\mathbf{\omega}^A \times \mathcal{F}\mathbf{v} = \Omega\mathbf{e}_z \times [(\dot{x} - \Omega\dot{y})\mathbf{e}_x + (\dot{y} + \Omega\dot{x})\mathbf{e}_y + \dot{z}\mathbf{e}_z] \]

\[
= \Omega(\ddot{x} - \Omega\dot{y})\mathbf{e}_y - \Omega(\ddot{y} + \Omega\dot{x})\mathbf{e}_x \quad (2.54)
\]

Adding Eqs. (2.53) and (2.54), we obtain \(\mathcal{F}\mathbf{a}\) as

\[
\mathcal{F}\mathbf{a} = (\dddot{x} - 2\Omega\ddot{y} - \Omega^2 x)\mathbf{e}_x + (\ddot{y} + 2\Omega\dot{x} - \Omega^2 y)\mathbf{e}_y + \ddot{z}\mathbf{e}_z \quad (2.55)
\]
Question 2–8

A bead slides along a fixed circular helix of radius $R$ and helical inclination angle $\phi$ as shown in Fig. P2-8. Knowing that the angle $\theta$ measures the position of the bead and is equal to zero when the bead is at the base of the helix, determine the following quantities relative to an observer fixed to the helix: (a) the arclength parameter $s$ as a function of the angle $\theta$, (b) the intrinsic basis \{\(e_t\), \(e_n\), \(e_b\)\} and the curvature of the trajectory as a function of the angle $\theta$, and (c) the position, velocity, and acceleration of the particle in terms of the intrinsic basis \{\(e_t\), \(e_n\), \(e_b\)\}.

![Figure P2-8](image)

Solution to Question 2–8

Let $F$ be a reference frame fixed to the helix. Then, choose the following coordinate system fixed in reference frame $F$:

<table>
<thead>
<tr>
<th>Basis Vector</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$E_x$</td>
<td>Along $OA$</td>
</tr>
<tr>
<td>$E_z$</td>
<td>Out of page</td>
</tr>
<tr>
<td>$E_y$</td>
<td>$E_z \times E_x$</td>
</tr>
</tbody>
</table>

Next, let $A$ be a reference frame that rotates with the projection of the position of particle into the \\{\(E_x\), \(E_y\)\}-plane. Corresponding to $A$, we choose the
following coordinate system to describe the motion of the particle:

\[
\begin{align*}
\mathbf{e}_r &= \text{Along } O \text{ to projection of } P \text{ into } \{E_x, E_y\} \text{ plane} \\
\mathbf{e}_z &= E_z \\
\mathbf{e}_\theta &= \mathbf{e}_z \times \mathbf{e}_r
\end{align*}
\]

Now, since \( \phi \) is the angle formed by the helix with the horizontal, we have from the geometry that

\[
z = R \theta \tan \phi \quad (2.56)
\]

Then the position of the bead can be written as

\[
\mathbf{r} = R \mathbf{e}_r + \tan \phi R \mathbf{e}_z = R \mathbf{e}_r + R \theta \tan \phi \mathbf{e}_z \quad (2.57)
\]

Furthermore, the angular velocity of reference frame \( \mathcal{A} \) in reference frame \( \mathcal{F} \) is given as

\[
\mathcal{F} \omega^\mathcal{A} = \dot{\theta} \mathbf{e}_z \quad (2.58)
\]

Then, applying the rate of change transport theorem to \( \mathbf{r} \) between reference frames \( \mathcal{A} \) and \( \mathcal{F} \), we have

\[
\mathcal{F} \mathbf{v} = \mathcal{F} \frac{d\mathbf{r}}{dt} = \mathcal{A} \frac{d\mathbf{r}}{dt} + \mathcal{F} \omega^\mathcal{A} \times \mathbf{r} \quad (2.59)
\]

where

\[
\begin{align*}
\mathcal{A} \frac{d\mathbf{r}}{dt} &= R \dot{\theta} \tan \phi \mathbf{e}_z \\
\mathcal{F} \omega^\mathcal{A} \times \mathbf{r} &= \dot{\theta} \mathbf{e}_z \times (R \mathbf{e}_r + R \theta \tan \phi \mathbf{e}_z) = R \dot{\theta} \mathbf{e}_\theta
\end{align*}
\]

Adding Eqs. (2.60) and (2.61), we obtain

\[
\mathcal{F} \mathbf{v} = R \dot{\theta} \mathbf{e}_\theta + R \dot{\theta} \tan \phi \mathbf{e}_z \quad (2.62)
\]

The speed in reference frame \( \mathcal{F} \) is then given as

\[
\mathcal{F} \mathbf{v} = \|\mathcal{F} \mathbf{v}\| = R \dot{\theta} \sqrt{1 + \tan^2 \phi} = R \dot{\theta} \sec \phi \equiv \frac{d}{dt} (\mathcal{F} s) \quad (2.63)
\]

Consequently,

\[
\mathcal{F} ds = R \sec \phi d\theta \quad (2.64)
\]

Integrating both sides of Eq. (2.64), we obtain

\[
\int_{s_0}^{s} ds = \int_{\theta_0}^{\theta} R \sec \phi d\theta \quad (2.65)
\]

We then obtain

\[
\mathcal{F}_s - \mathcal{F}_{s_0} = R (\theta - \theta_0) \sec \phi \quad (2.66)
\]

Solving Eq. (2.66) for \( s \), the arclength is given as

\[
\mathcal{F}_s = \mathcal{F}_{s_0} + R (\theta - \theta_0) \sec \phi \quad (2.67)
\]
Intrinsic Basis and Curvature of Trajectory

The intrinsic basis is obtained as follows. First, the tangent vector $e_t$ is given as

$$e_t = \frac{\mathcal{F}_V}{\mathcal{F}_V}$$  \hspace{1cm} (2.68)

Substituting the expressions for $\mathcal{F}_V$ and $\mathcal{F}_V$ from part (a) into Eq. (2.68), we obtain

$$e_t = \frac{R \dot{\theta} e_\theta + R \dot{\theta} \tan \phi e_z}{R \dot{\theta} \sec \phi} = \cos \phi e_\theta + \sin \phi e_z$$  \hspace{1cm} (2.69)

Next, we have

$$\mathcal{F} \frac{de_t}{dt} = \mathcal{A} \frac{de_t}{dt} + \mathcal{F} \omega^A \times e_t$$  \hspace{1cm} (2.70)

where

$$\mathcal{A} \frac{de_t}{dt} = 0$$  \hspace{1cm} (2.71)

$$\mathcal{F} \omega^A \times e_t = \dot{\theta} e_z \times (\cos \phi e_\theta + \sin \phi e_z) = -\dot{\theta} \cos \phi e_r$$

Therefore,

$$\mathcal{F} \frac{de_t}{dt} = -\dot{\theta} \cos \phi e_r$$  \hspace{1cm} (2.72)

The principle unit normal is then given as

$$e_n = \frac{\mathcal{F} \frac{de_t}{dt}}{|| \mathcal{F} \frac{de_t}{dt} ||} = -e_r$$  \hspace{1cm} (2.73)

Finally, the principle unit bi-normal vector is given as

$$e_b = e_t \times e_n = (\cos \phi e_\theta + \sin \phi e_z) \times (-e_r) = -\sin \phi e_\theta + \cos \phi e_z$$  \hspace{1cm} (2.74)

Position, Velocity, and Acceleration of Bead

First, we can solve for the basis $\{e_r, e_\theta, e_z\}$ in terms of $\{e_t, e_n, e_b\}$ by using Eqs. (2.69), (2.73), and (2.74). First, from Eq. (2.73), we have

$$e_r = -e_n$$  \hspace{1cm} (2.75)

Next, restating Eqs. (2.69) and (2.74), we have

$$e_t = \cos \phi e_\theta + \sin \phi e_z$$  \hspace{1cm} (2.76)

$$e_b = -\sin \phi e_\theta + \cos \phi e_z$$  \hspace{1cm} (2.77)

Solving Eqs. (2.76) and (2.77) simultaneously for $e_\theta$ and $e_z$, we obtain

$$e_\theta = \sin \phi e_t + \cos \phi e_b$$  \hspace{1cm} (2.78)

$$e_z = \cos \phi e_t - \sin \phi e_b$$  \hspace{1cm} (2.79)
Then, substituting the expressions for $e_r$ and $e_z$ from Eqs. (2.75) and (2.79) into Eq. (2.57), we have the position in terms of $\{e_t, e_n, e_b\}$ as

$$r = -Re_n + R\theta \tan \phi (\cos \phi e_t - \sin \phi e_b)$$  \hspace{1cm} (2.80)

Next, the velocity in reference frame $F$ is given in terms of $\{e_t, e_n, e_b\}$ as

$$\mathcal{F}v = \mathcal{F}v e_t$$  \hspace{1cm} (2.81)

Substituting the expression for $\mathcal{F}v$ from Eq. (2.63) into Eq. (2.81), we have

$$\mathcal{F}v = R\dot{\theta} \sec \phi e_t$$  \hspace{1cm} (2.82)

Finally, the acceleration in reference frame $F$ is given as

$$\mathcal{F}a = \frac{d}{dt} (\mathcal{F}v) e_t + \mathcal{F}v \left\| \frac{d}{dt} e_t \right\| e_n$$  \hspace{1cm} (2.83)

Computing the rate of change of $\mathcal{F}v$ using the expression for $\mathcal{F}v$ from Eq. (2.63), we have

$$\frac{d}{dt} (\mathcal{F}v) = R\ddot{\theta} \sec \phi$$  \hspace{1cm} (2.84)

Therefore,

$$\mathcal{F}a = R\ddot{\theta} \sec \phi e_t + R\dot{\theta} \sec \phi \dot{\theta} \cos \phi e_n = R\ddot{\theta} \sec \phi e_t + R\dot{\theta}^2 e_n$$  \hspace{1cm} (2.85)
Question 2–9

Arm $AB$ is hinged at points $A$ and $B$ to collars that slide along vertical and horizontal shafts, respectively, as shown in Fig. P2-9. The vertical shaft rotates with angular velocity $\Omega$ relative to a fixed reference frame (where $\Omega = \|\Omega\|$) and point $B$ moves with constant velocity $v_0$ relative to the horizontal shaft. Knowing that point $P$ is located at the center of the arm and the angle $\theta$ describes the orientation of the arm with respect to the vertical shaft, determine the velocity and acceleration of point $P$ as viewed by an observer fixed to the ground. In simplifying your answers, find an expression for $\dot{\theta}$ in terms of $v_0$ and $l$ and express your answers in terms of only $l$, $\Omega$, $\dot{\Omega}$, $\theta$, and $v_0$.

![Figure P2-9]

Solution to Question 2–9

Let $\mathcal{F}$ be the ground. Then choose the following coordinate system fixed in reference frame $\mathcal{F}$:

- Origin at $O$
- $E_x = \text{Along } OB \text{ when } t = 0$
- $E_y = \text{Along } OA$
- $E_z = E_x \times E_y$
Next, let $\mathcal{A}$ be the L-shaped assembly. Then choose the following coordinate system fixed in $\mathcal{A}$:

- Origin at $O$
- $e_x = $ Along $OB$
- $e_y = $ Along $OA$
- $e_z = e_x \times e_y$

Finally, let $\mathcal{B}$ be the rod. Then choose the following coordinate system fixed in $\mathcal{B}$:

- Origin at $A$
- $u_r = $ Along $AB$
- $u_z = e_z$
- $u_{\theta} = u_z \times u_r$

From the geometry of the coordinate systems, we have

\[
\begin{align*}
e_x & = \sin \theta u_r + \cos \theta u_{\theta} \\
e_y & = -\cos \theta u_r + \sin \theta u_{\theta}
\end{align*}
\]

(2.86)

Next, because we must measure all distances from point $O$ (because point $O$ is fixed to the ground and we want all rates of change as viewed by an observer fixed to the ground), the position of the center of the rod is given as

\[
r_{P/O} = r_{A/O} + r_{P/A} \equiv r
\]

(2.87)

Using the coordinates systems defined for this problem, we have

\[
\begin{align*}
r_{A/O} & = l \cos \theta e_y \\
r_{P/A} & = \frac{l}{2} u_r
\end{align*}
\]

(2.88)

Consequently,

\[
r_{P/O} = l \cos \theta e_y + \frac{1}{2} u_r
\]

(2.89)

Because $r_{A/O}$ is expressed in the basis $\{e_x, e_y, e_z\}$ while $r_{P/A}$ is expressed in the basis $\{u_r, u_{\theta}, u_z\}$, it is convenient to differentiate each piece of the vector $r_{P/O}$ separately. First, the velocity of point $A$ relative to point $O$ as viewed by an observer fixed to the ground is obtained by applying the transport theorem from $\mathcal{A}$ to $\mathcal{F}$ as

\[
\mathcal{F} v_{A/O} = \mathcal{F} \frac{d}{dt} (r_{A/O}) = \mathcal{A} \frac{d}{dt} (r_{A/O}) + \mathcal{F} \omega^{\mathcal{A}} \times r_{A/O}
\]

(2.90)

First, we have

\[
\mathcal{F} \omega^{\mathcal{A}} = \Omega = \Omega e_y
\]

(2.91)

Next,

\[
\begin{align*}
\mathcal{A} \frac{d}{dt} (r_{A/O}) & = -l \dot{\theta} \sin \theta e_y \\
\mathcal{F} \omega^{\mathcal{A}} \times r_{A/O} & = \Omega e_y \times l \cos \theta e_y = 0
\end{align*}
\]

(2.92)
Consequently,
\[ F_{v_{A/O}} = -l\dot{\theta} \sin \theta e_y \] (2.93)

The acceleration of point A relative to point O as viewed by an observer fixed to the ground is then given as
\[ F_{a_{A/O}} = \frac{\partial}{\partial t} (F_{v_{A/O}}) = \frac{A}{\partial t} (F_{v_{A/O}}) + F_\omega A \times F_{v_{A/O}} \] (2.94)

Now we have
\[ \frac{A}{\partial t} (F_{v_{A/O}}) = -l(\ddot{\theta} \sin \theta + \dot{\theta}^2 \cos \theta) e_y \]
\[ F_\omega A \times F_{v_{A/O}} = \Omega e_y \times (-l\dot{\theta} \sin \theta) e_y = 0 \] (2.95)

Therefore,
\[ F_{a_{A/O}} = -l(\ddot{\theta} \sin \theta + \dot{\theta}^2 \cos \theta) e_y \] (2.96)

The velocity of point P relative to point A as viewed by an observer fixed to the ground is obtained by applying the transport theorem from reference frame \( B \) to reference frame \( F \) as
\[ F_{v_{P/A}} = \frac{\partial}{\partial t} (F_{v_{P/A}}) = \frac{B}{\partial t} (F_{v_{P/A}}) + F_\omega B \times F_{v_{P/A}} \] (2.97)

Now
\[ F_\omega B = A_\omega B \] (2.98)

where
\[ A_\omega B = \dot{\theta} u_z \] (2.99)

Therefore,
\[ F_\omega B = \Omega e_y + \dot{\theta} u_z = \Omega (-\cos \theta u_r + \sin \theta u_\theta) + \dot{\theta} u_z \]
\[ = -\Omega \cos \theta u_r + \Omega \sin \theta u_\theta + \dot{\theta} u_z \] (2.100)

Now we have
\[ \frac{B}{\partial t} (F_{r_{P/A}}) = 0 \]
\[ F_\omega B \times F_{r_{P/A}} = (-\Omega \cos \theta u_r + \Omega \sin \theta u_\theta + \dot{\theta} u_z) \times \frac{l}{2} u_r \]
\[ = \frac{l}{2} u_\theta - \frac{\Omega \sin \theta}{2} u_z \] (2.101)

Therefore,
\[ F_{v_{P/A}} = \frac{l}{2} u_\theta - \frac{l}{2} \Omega \sin \theta u_z \] (2.102)

The acceleration of point P relative to point A as viewed by an observer fixed to the ground is then given as
\[ F_{a_{P/A}} = \frac{\partial}{\partial t} (F_{v_{P/A}}) = \frac{B}{\partial t} (F_{v_{P/A}}) + F_\omega B \times F_{v_{P/A}} \] (2.103)
Now we have
\[
\frac{d}{dt} \left( J \mathbf{v}_{P/A} \right) = \frac{l \dot{\theta}}{2} \mathbf{u}_\theta - \frac{l \Omega \dot{\theta} \cos \theta}{2} \mathbf{u}_z
\]

\[
J \mathbf{w}^B \times J \mathbf{v}_{P/A} = \left( -\Omega \cos \theta \mathbf{u}_r + \Omega \sin \theta \mathbf{u}_\theta + \dot{\theta} \mathbf{u}_z \right) \times \left( \frac{l \dot{\theta}}{2} \mathbf{u}_\theta - \frac{l \Omega \sin \theta}{2} \mathbf{u}_z \right)
\]  
(2.104)

The second term in Eq. (2.104) can be simplified to
\[
J \mathbf{w}^B \times J \mathbf{v}_{P/A} = -\frac{l \Omega \dot{\theta} \cos \theta}{2} \mathbf{u}_z - \frac{l \Omega^2 \cos \theta \sin \theta}{2} \mathbf{u}_\theta - \left( \frac{l \Omega^2 \sin^2 \theta + \dot{\theta}^2}{2} \right) \mathbf{u}_r
\]  
(2.105)

Adding the first term in Eq. (2.104) to the result of Eq. (2.105), we obtain the acceleration of point \( P \) relative to point \( A \) as viewed by an observer fixed to the ground as
\[
J \mathbf{a}_{P/A} = -\left( \frac{l \Omega^2 \sin^2 \theta + \dot{\theta}^2}{2} \right) \mathbf{u}_r + \left( \frac{l \dot{\theta}}{2} - \frac{l \Omega^2 \cos \theta \sin \theta}{2} \right) \mathbf{u}_\theta - l \dot{\Omega} \dot{\theta} \cos \theta \mathbf{u}_z
\]  
(2.106)

Using the aforementioned results, we obtain the velocity and acceleration of point \( P \) relative to point \( O \) as viewed by an observer fixed to the ground as follows. First, adding the results of Eqs. (2.93) and (2.102), we obtain
\[
J \mathbf{v}_{P/O} = -l \dot{\theta} \sin \theta \mathbf{e}_y + \frac{l \dot{\theta}}{2} \mathbf{u}_\theta - \frac{l \Omega \sin \theta}{2} \mathbf{u}_z
\]  
(2.107)

Finally, adding the results of Eqs. (2.96) and (2.106), we obtain
\[
J \mathbf{a}_{P/O} = -l (\dot{\theta} \sin \theta + \dot{\theta}^2 \cos \theta) \mathbf{e}_y - \left( \frac{l \Omega^2 \sin^2 \theta + \dot{\theta}^2}{2} \right) \mathbf{u}_r + \left( \frac{l \dot{\theta}}{2} - \frac{l \Omega^2 \cos \theta \sin \theta}{2} \right) \mathbf{u}_\theta - l \dot{\Omega} \dot{\theta} \cos \theta \mathbf{u}_z
\]  
(2.108)

A last point pertains to the velocity of point \( B \). It was stated in the problem that, “point \( B \) moves with constant velocity \( \mathbf{v}_0 \) relative to the horizontal shaft.” Now, because the horizontal shaft is fixed in reference frame \( A \), we have
\[
A \mathbf{v}_B = \mathbf{v}_0 \mathbf{e}_x = \text{constant}
\]  
(2.109)

Another expression for the \( A \mathbf{v}_B \) is obtained as follows. First,
\[
\mathbf{r}_B = l \sin \theta \mathbf{e}_x
\]  
(2.110)

Therefore,
\[
A \mathbf{v}_B = l \dot{\theta} \cos \theta \mathbf{e}_x
\]  
(2.111)
Equating the expressions in Eq. (2.109) and (2.111), we obtain

\[ v_0 = l \dot{\theta} \cos \theta \]  

(2.112)

which implies that

\[ \dot{\theta} = \frac{v_0}{l \cos \theta} = \frac{v_0}{l} \sec \theta \]  

(2.113)

Differentiating Eq. (2.113) with respect to time, we have

\[ \ddot{\theta} = \frac{v_0}{l} \dot{\theta} \sec \theta \tan \theta = \frac{v_0^2}{l^2} \sec^2 \theta \tan \theta \]  

(2.114)

The expressions for \( \dot{\theta} \) and \( \ddot{\theta} \) given in Eqs. (2.113) and (2.114), respectively, can be substituted into the expressions for \( \mathcal{F} v_{P/O} \) and \( \mathcal{F} a_{P/O} \) to obtain expressions that do not involve either \( \dot{\theta} \) or \( \ddot{\theta} \).
Question 2–10

A circular disk of radius $R$ is attached to a rotating shaft of length $L$ as shown in Fig. P2-10. The shaft rotates about the horizontal direction with a constant angular velocity $\Omega$ relative to the ground. The disk, in turn, rotates about its center about an axis orthogonal to the shaft. Knowing that the angle $\theta$ describes the position of a point $P$ located on the edge of the disk relative to the center of the disk, determine the velocity and acceleration of point $P$ relative to the ground.

Solution to Question 2–10

First, let $\mathcal{F}$ be a reference frame fixed to the ground. Then, we choose the following coordinate system fixed in reference frame $\mathcal{F}$:

<table>
<thead>
<tr>
<th>Basis Vector</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$E_2$</td>
<td>Along $AO$</td>
</tr>
<tr>
<td>$E_3$</td>
<td>Orthogonal to Disk and Into Page at $t = 0$</td>
</tr>
<tr>
<td>$E_1$</td>
<td>$E_2 \times E_3$</td>
</tr>
</tbody>
</table>

Next, let $\mathcal{A}$ be a reference frame fixed to the horizontal shaft. Then, we choose the following coordinate system fixed in reference frame $\mathcal{F}$:

<table>
<thead>
<tr>
<th>Basis Vector</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$e_2$</td>
<td>Along $AO$</td>
</tr>
<tr>
<td>$e_3$</td>
<td>Orthogonal to Disk and Into Page</td>
</tr>
<tr>
<td>$e_1$</td>
<td>$e_2 \times e_3$</td>
</tr>
</tbody>
</table>
Lastly, let $\mathcal{B}$ be a reference frame fixed to the disk. Then, choose the following coordinate system fixed in reference frame $\mathcal{B}$:

- **Origin at Point O**
  - $u_1 = \text{Along } OP$
  - $u_3 = \text{Orthogonal to Disk and Into Page}$
  - $u_2 = u_3 \times u_1$

Now, since the shaft rotates with angular velocity $\Omega$ relative to the ground, the angular velocity of reference frame $\mathcal{A}$ in reference frame $\mathcal{F}$ is given as

$$\mathcal{F} \omega^A = \Omega = \Omega e_2$$  \hspace{1cm} (2.115)

Furthermore, since the disk rotates with angular rate $\dot{\theta}$ relative to the shaft, the angular velocity of reference frame $\mathcal{B}$ in reference frame $\mathcal{A}$ is given as

$$\mathcal{A} \omega^B = \dot{\theta} u_3$$  \hspace{1cm} (2.116)

Finally, the geometry of the bases $\{e_1, e_2, e_3\}$ and $\{u_1, u_2, u_3\}$ is shown in Fig. (2.117). Using Fig. (2.117), we have that

$$e_1 = \cos \theta u_1 - \sin \theta u_2$$
$$e_2 = \sin \theta u_1 + \cos \theta u_2$$  \hspace{1cm} (2.117)

![Figure 2-2](image)

**Figure 2-2**  Relationship Between Basis $\{e_1, e_2, e_3\}$ and $\{u_1, u_2, u_3\}$ for Question 2-10

Given the definitions of the reference frames and coordinate systems, the position of point $P$ is given as

$$r = Ru_1$$  \hspace{1cm} (2.118)
The velocity of point $P$ in reference frame $F$ is then given as
\[ F\mathbf{v} = \frac{F\mathbf{d}}{dt} = \frac{F\mathbf{d}}{dt} (R\mathbf{u}_1) \]  
(2.119)

Now, since the basis $\{u_1, u_2, u_3\}$ is fixed in reference frame $F$, it is convenient to apply the rate of change transport theorem of Eq. (2–128) between reference frame $B$ and reference frame $F$ as
\[ \frac{F\mathbf{d}}{dt} (R\mathbf{u}_1) = \frac{B\mathbf{d}}{dt} (R\mathbf{u}_1) + F \omega^B \times (R\mathbf{u}_1) \]  
(2.120)

First, since $R$ is constant and the basis $\{u_1, u_2, u_3\}$ is fixed in reference frame $B$, we have that
\[ \frac{B\mathbf{d}}{dt} (R\mathbf{u}_1) = 0 \]  
(2.121)

Next, applying the angular velocity addition rule of Eq. (2–136), we obtain $F \omega^B$ as
\[ F \omega^B = F \omega^A + A \omega^B = \Omega \mathbf{e}_2 + \dot{\theta} \mathbf{u}_3 \]  
(2.122)

Using $F \omega^B$ from Eq. (2.122), we obtain $F \omega^B \times R\mathbf{u}_1$ as
\[ F \omega^B \times R\mathbf{u}_1 = (\Omega \mathbf{e}_2 + \dot{\theta} \mathbf{u}_3) \times R\mathbf{u}_1 = R\Omega \mathbf{e}_2 \times \mathbf{u}_1 + R \dot{\theta} \mathbf{u}_2 \]  
(2.123)

Then, from Eq. (2.117), we have that
\[ \mathbf{e}_2 \times \mathbf{u}_1 = (\sin \theta \mathbf{u}_1 + \cos \theta \mathbf{u}_2) \times \mathbf{u}_1 = -\cos \theta \mathbf{u}_3 \]  
(2.124)

Substituting the result of Eq. (2.124) into Eq. (2.123), we obtain
\[ F \omega^B \times R\mathbf{u}_1 = -R\Omega \cos \theta \mathbf{u}_3 + R \dot{\theta} \mathbf{u}_2 \]  
(2.125)

Adding Eq. (2.121) and Eq. (2.125), we obtain the velocity of point $P$ in reference frame $F$ as
\[ F\mathbf{v} = R \dot{\theta} \mathbf{u}_2 - R\Omega \cos \theta \mathbf{u}_3 \]  
(2.126)

Next, the acceleration of point $P$ in reference frame $F$ is given as
\[ F\mathbf{a} = \frac{F\mathbf{d}}{dt} (F\mathbf{v}) \]  
(2.127)

It is seen that the expression for $F\mathbf{v}$ is given in terms of the basis $\{u_1, u_2, u_3\}$ where $\{u_1, u_2, u_3\}$ is fixed in reference frame $B$. Thus, applying the rate of change transport theorem of Eq. (2–128) between reference frame $B$ and $F$ to $F\mathbf{v}$, we obtain
\[ F\mathbf{a} = \frac{F\mathbf{d}}{dt} (F\mathbf{v}) = \frac{B\mathbf{d}}{dt} (F\mathbf{v}) + F \omega^B \times F\mathbf{v} \]  
(2.128)
Now, observing that $R$ and $\Omega$ are constant, the first term in Eq. (2.128) is given as
\[
\frac{\mathbf{b}}{\mathbf{d}} \left( \mathcal{F} \mathbf{v} \right) = R \ddot{\theta} \mathbf{u}_2 + R \Omega \dot{\theta} \sin \theta \mathbf{u}_3
\] (2.129)

Next, using $\mathcal{F} \mathbf{B}$ from Eq. (2.122), we obtain the second term in Eq. (2.128) as
\[
\mathcal{F} \mathbf{B} \times \mathcal{F} \mathbf{v} = (\Omega \mathbf{e}_2 + \dot{\theta} \mathbf{u}_3) \times (-R \Omega \cos \theta \mathbf{u}_3 + R \dot{\theta} \mathbf{u}_2)
\] (2.130)

Expanding Eq. (2.130), we obtain
\[
\mathcal{F} \mathbf{B} \times \mathcal{F} \mathbf{v} = R \Omega \dot{\theta} \mathbf{e}_2 \times \mathbf{u}_2 - R \Omega^2 \cos \theta \mathbf{e}_2 \times \mathbf{u}_3 - R \dot{\theta}^2 \mathbf{u}_1
\] (2.131)

Then, using the expression for $\mathbf{e}_2$ from Eq. (2.117), we obtain
\[
\begin{align*}
\mathbf{e}_2 \times \mathbf{u}_2 &= (\sin \theta \mathbf{u}_1 + \cos \theta \mathbf{u}_2) \times \mathbf{u}_2 = \sin \theta \mathbf{u}_3 \\
\mathbf{e}_2 \times \mathbf{u}_3 &= (\sin \theta \mathbf{u}_1 + \cos \theta \mathbf{u}_2) \times \mathbf{u}_3 = \cos \theta \mathbf{u}_1 - \sin \theta \mathbf{u}_2
\end{align*}
\] (2.132)

Substituting the results of Eq. (2.132) into Eq. (2.131), we obtain
\[
\mathcal{F} \mathbf{B} \times \mathcal{F} \mathbf{v} = R \Omega \dot{\theta} \sin \theta \mathbf{u}_3 - R \Omega^2 \cos \theta (\cos \theta \mathbf{u}_1 - \sin \theta \mathbf{u}_2) - R \dot{\theta}^2 \mathbf{u}_1
\] (2.133)

Adding the expressions in Eq. (2.129) and Eq. (2.133), we obtain the acceleration of point $P$ in reference frame $\mathcal{F}$ as
\[
\mathcal{F} \mathbf{a} = R \ddot{\theta} \mathbf{u}_2 + R \Omega \dot{\theta} \sin \theta \mathbf{u}_3 + R \Omega \dot{\theta} \sin \theta \mathbf{u}_3 - R \Omega^2 \cos \theta (\cos \theta \mathbf{u}_1 - \sin \theta \mathbf{u}_2) - R \dot{\theta}^2 \mathbf{u}_1
\] (2.134)

Simplifying Eq. (2.134), we obtain
\[
\mathcal{F} \mathbf{a} = -(R \Omega^2 \cos^2 \theta + R \dot{\theta}^2) \mathbf{u}_1 + (R \ddot{\theta} + R \Omega^2 \cos \theta \sin \theta) \mathbf{u}_2 + 2R \Omega \dot{\theta} \sin \theta \mathbf{u}_3
\] (2.135)