Question 2–11

A rod of length L with a wheel of radius R attached to one of its ends is rotating about the vertical axis OA with a constant angular velocity $\mathbf{\Omega}$ relative to a fixed reference frame as shown in Fig. P2-11. The wheel is vertical and rolls without slip along a fixed horizontal surface. Determine the angular velocity and angular acceleration of the wheel as viewed by an observer in a fixed reference frame.

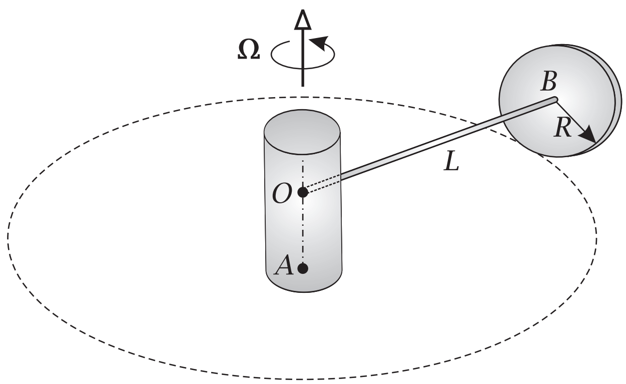

Figure P2-11

Solution to Question 2–11

Let $\mathcal{F}$ be a reference frame fixed to the ground. Then choose the following coordinate system fixed in reference frame $\mathcal{F}$:

$$
\begin{array}{lcl}
 & \text{Origin at } O & \\
\mathbf{E}_x & = & \text{Along } OB \text{ when } t = 0 \\
\mathbf{E}_z & = & \text{Along } \mathbf{\Omega} \\
\mathbf{E}_y & = & \mathbf{E}_z \times \mathbf{E}_x
\end{array}
$$

Next, let $\mathcal{A}$ be a reference frame fixed to the direction of OB. Then choose the following coordinate system fixed in reference frame $\mathcal{A}$:

$$
\begin{array}{lcl}
 & \text{Origin at } O & \\
\mathbf{e}_x & = & \text{Along } OB \\
\mathbf{e}_z & = & \text{Along } \mathbf{\Omega} \\
\mathbf{e}_y & = & \mathbf{e}_z \times \mathbf{e}_x
\end{array}
$$

Finally, let $\mathcal{D}$ be the reference frame of the wheel. Then choose the following coordinate system fixed in reference frame $\mathcal{D}$:

$$
\begin{array}{lcl}
 & \text{Origin at } B & \\
\mathbf{u}_x & = & \text{Along } OB \\
\mathbf{u}_y & = & \text{In the plane of the wheel} \\
\mathbf{u}_z & = & \text{In the plane of the wheel} = \mathbf{u}_x \times \mathbf{u}_y
\end{array}
$$

Now the angular velocity of the arm OB as viewed by an observer fixed to the ground, denoted ${}^{\mathcal{F}}\boldsymbol{\omega}^{\mathcal{A}}$, is given as

$$ {}^{\mathcal{F}}\boldsymbol{\omega}^{\mathcal{A}} = \boldsymbol{\Omega} = \Omega \mathbf{e}_z \tag{2.136} $$

Next, the position of point B is given as

$$ \mathbf{r}_B = L\mathbf{e}_x \tag{2.137} $$

Computing the rate of change of $\mathbf{r}_B$ in reference frame $\mathcal{F}$, we obtain the velocity of point B in reference frame $\mathcal{F}$ as

$$ {}^{\mathcal{F}}\mathbf{v}_B = \frac{{}^{\mathcal{F}}d\mathbf{r}_B}{dt} = \frac{{}^{\mathcal{A}}d\mathbf{r}_B}{dt} + {}^{\mathcal{F}}\boldsymbol{\omega}^{\mathcal{A}} \times \mathbf{r}_B \tag{2.138} $$

Now we have

$$ \begin{aligned} \frac{{}^{\mathcal{A}}d\mathbf{r}_B}{dt} &= \mathbf{0} && (2.139) \\ {}^{\mathcal{F}}\boldsymbol{\omega}^{\mathcal{A}} \times \mathbf{r}_B &= \Omega \mathbf{e}_z \times L\mathbf{e}_x = L\Omega \mathbf{e}_y && (2.140) \end{aligned} $$

Consequently,

$$ {}^{\mathcal{F}}\mathbf{v}_B = L\Omega \mathbf{e}_y \tag{2.141} $$

Next, suppose we let Q be the point of contact of the wheel with the ground. Then, because the wheel rolls without slip along the ground, we know that

$$ {}^{\mathcal{F}}\mathbf{v}_Q^{\mathcal{D}} = \mathbf{0} \tag{2.142} $$

Then, using Eq. (2–517) on page 107, we can obtain a second expression for ${}^{\mathcal{F}}\mathbf{v}_B$ as

$$ {}^{\mathcal{F}}\mathbf{v}_B = {}^{\mathcal{F}}\mathbf{v}_Q^{\mathcal{D}} + {}^{\mathcal{F}}\boldsymbol{\omega}^{\mathcal{D}} \times (\mathbf{r}_B - \mathbf{r}_Q) \tag{2.143} $$

Now we know that ${}^{\mathcal{F}}\boldsymbol{\omega}^{\mathcal{D}}$ is given from the angular velocity addition theorem as

$$ {}^{\mathcal{F}}\boldsymbol{\omega}^{\mathcal{D}} = {}^{\mathcal{F}}\boldsymbol{\omega}^{\mathcal{A}} + {}^{\mathcal{A}}\boldsymbol{\omega}^{\mathcal{D}} \tag{2.144} $$

We already have ${}^{\mathcal{F}}\boldsymbol{\omega}^{\mathcal{A}}$ from earlier. Then, because the wheel rotates about the $\mathbf{e}_x$-direction ($\equiv \mathbf{u}_x$-direction) and $\mathbf{e}_x = \mathbf{u}_x$ is fixed in reference frame $\mathcal{A}$, we have

$$ {}^{\mathcal{A}}\boldsymbol{\omega}^{\mathcal{D}} = \omega \mathbf{u}_x \tag{2.145} $$

where ω is to be determined. Adding Eqs. (2.136) and (2.145), we obtain

$$ {}^{\mathcal{F}}\boldsymbol{\omega}^{\mathcal{D}} = \Omega \mathbf{e}_z + \omega \mathbf{u}_x \tag{2.146} $$

Also, $\mathbf{r}_B - \mathbf{r}_Q$ is given as

$$ \mathbf{r}_B - \mathbf{r}_Q = R\mathbf{E}_z = R\mathbf{e}_z \tag{2.147} $$

Therefore,

$$ {}^{\mathcal{F}}\mathbf{v}_B = (\Omega \mathbf{e}_z + \omega \mathbf{e}_x) \times R\mathbf{e}_z = -R\omega \mathbf{e}_y \tag{2.148} $$

Setting the expressions for ${}^{\mathcal{F}}\mathbf{v}_B$ from Eqs. (2.148) and (2.148) equal, we obtain

$$ L\Omega = -R\omega \tag{2.149} $$

from which we obtain ω as

$$ \omega = -\frac{L}{R}\Omega \tag{2.150} $$

The angular velocity of the wheel as viewed by an observer fixed to the ground is then given as

$$ {}^{\mathcal{D}}\boldsymbol{\omega}^{\mathcal{F}} = \Omega \mathbf{e}_z - \frac{L}{R}\Omega \mathbf{e}_x \tag{2.151} $$

The angular acceleration of the wheel in reference frame $\mathcal{F}$ is then given as

$$ {}^{\mathcal{F}}\boldsymbol{\alpha}^{\mathcal{D}} = \frac{{}^{\mathcal{F}}d}{dt}\left({}^{\mathcal{F}}\boldsymbol{\omega}^{\mathcal{D}}\right) = \frac{{}^{\mathcal{F}}d}{dt}\left({}^{\mathcal{F}}\boldsymbol{\omega}^{\mathcal{A}}\right) + \frac{{}^{\mathcal{F}}d}{dt}\left({}^{\mathcal{A}}\boldsymbol{\omega}^{\mathcal{D}}\right) \tag{2.152} $$

Now, because ${}^{\mathcal{F}}\boldsymbol{\omega}^{\mathcal{A}} = \Omega \mathbf{e}_z = \Omega \mathbf{E}_z$ and $\mathbf{E}_z$ is fixed in reference frame $\mathcal{F}$, we have

$$ \frac{{}^{\mathcal{F}}d}{dt}\left({}^{\mathcal{F}}\boldsymbol{\omega}^{\mathcal{A}}\right) = \dot{\Omega}\mathbf{e}_z = \mathbf{0} \tag{2.153} $$

because Ω is constant. Next, because ${}^{\mathcal{A}}\boldsymbol{\omega}^{\mathcal{D}} = -(L/R)\Omega \mathbf{e}_x$ and $\mathbf{e}_x$ is fixed in reference frame $\mathcal{A}$, we can apply the transport theorem to ${}^{\mathcal{A}}\boldsymbol{\omega}^{\mathcal{D}}$ between reference frames $\mathcal{A}$ and $\mathcal{F}$ as

$$ \frac{{}^{\mathcal{F}}d}{dt}\left({}^{\mathcal{A}}\boldsymbol{\omega}^{\mathcal{D}}\right) = \frac{{}^{\mathcal{A}}d}{dt}\left({}^{\mathcal{A}}\boldsymbol{\omega}^{\mathcal{D}}\right) + {}^{\mathcal{F}}\boldsymbol{\omega}^{\mathcal{A}} \times {}^{\mathcal{A}}\boldsymbol{\omega}^{\mathcal{D}} \tag{2.154} $$

Now we have

$$ \begin{aligned} \frac{{}^{\mathcal{A}}d}{dt}\left({}^{\mathcal{A}}\boldsymbol{\omega}^{\mathcal{D}}\right) &= -\frac{L}{R}\dot{\Omega}\mathbf{e}_x = \mathbf{0} && (2.155) \\ {}^{\mathcal{F}}\boldsymbol{\omega}^{\mathcal{A}} \times {}^{\mathcal{A}}\boldsymbol{\omega}^{\mathcal{D}} &= \Omega \mathbf{e}_z \times \left(-\frac{L}{R}\Omega \mathbf{e}_x\right) = -\frac{L}{R}\Omega^2 \mathbf{e}_y && (2.156) \end{aligned} $$

where we have again used the fact that Ω is constant. Therefore,

$$ \frac{{}^{\mathcal{F}}d}{dt}\left({}^{\mathcal{A}}\boldsymbol{\omega}^{\mathcal{D}}\right) = -\frac{L}{R}\Omega^2 \mathbf{e}_y \tag{2.157} $$

Consequently, the angular acceleration of the disk as viewed by an observer fixed to the ground is given as

$$ {}^{\mathcal{F}}\boldsymbol{\alpha}^{\mathcal{D}} = -\frac{L}{R}\Omega^2 \mathbf{e}_y \tag{2.158} $$

Question 2–13

A collar is constrained to slide along a track in the form of a logarithmic spiral as shown in Fig. P2-13. The equation for the spiral is given as

$$r = r_0 e^{-a\theta}$$

where r_0 and a are constants and θ is the angle measured from the horizontal direction. Determine (a) expressions for the intrinsic basis vectors $\mathbf{e}_t$, $\mathbf{e}_n$, and $\mathbf{e}_b$ in terms any other basis of your choosing, (b) the curvature of the trajectory as a function of the angle θ, and (c) the velocity and acceleration of the collar as viewed by an observer fixed to the track.

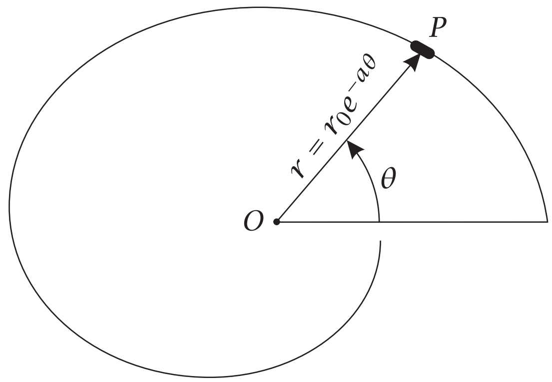

Figure P2-13

Solution to Question 2–13

(a) Intrinsic Basis

Let $\mathcal{F}$ be a reference frame fixed to the track. Then, choose the following coordinate system fixed in reference frame $\mathcal{F}$:

Origin at O
$\mathbf{E}_x$ To the Right
$\mathbf{E}_z$ Out of Page
$\mathbf{E}_y = \mathbf{E}_z \times \mathbf{E}_x$

Next, let $\mathcal{A}$ be a reference frame that rotates with the direction along Om. Then, choose the following coordinate system fixed in reference frame $\mathcal{A}$:

Origin at O
$\mathbf{e}_r$ Along Om
$\mathbf{E}_z$ Out of Page
$\mathbf{e}_\theta = \mathbf{E}_z \times \mathbf{e}_r$

The position of the particle is given in terms of the basis $\{\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_z\}$ as

$$\mathbf{r} = r\mathbf{e}_r = r_0 e^{-a\theta}\mathbf{e}_r \tag{2.159}$$

Furthermore, the angular velocity of reference frame $\mathcal{A}$ in reference frame $\mathcal{F}$ is given as

$$^{\mathcal{F}}\boldsymbol{\omega}^{\mathcal{A}} = \dot{\theta}\mathbf{E}_z \tag{2.160}$$

Applying the rate of change transport theorem between reference frame $\mathcal{A}$ and reference frame $\mathcal{F}$, the velocity of the particle in reference frame $\mathcal{F}$ is given as

$$^{\mathcal{F}}\mathbf{v} = \frac{^{\mathcal{F}}d\mathbf{r}}{dt} = \frac{^{\mathcal{A}}d\mathbf{r}}{dt} + {^{\mathcal{F}}\boldsymbol{\omega}^{\mathcal{A}}} \times \mathbf{r} \tag{2.161}$$

where

$$\begin{aligned} \frac{^{\mathcal{A}}d\mathbf{r}}{dt} &= \dot{r}\mathbf{e}_r = -ar_0\dot{\theta}e^{-a\theta}\mathbf{e}_r && (2.162) \\ {^{\mathcal{F}}\boldsymbol{\omega}^{\mathcal{A}}} \times \mathbf{r} &= \dot{\theta}\mathbf{e}_z \times r\mathbf{e}_r = \dot{\theta}\mathbf{e}_z \times r_0 e^{-a\theta}\mathbf{e}_r = r_0\dot{\theta}e^{-a\theta}\mathbf{e}_\theta && (2.163) \end{aligned}$$

Adding Eq. (2.162) and Eq. (2.163), we obtain the velocity of the particle in reference frame $\mathcal{F}$ as

$$^{\mathcal{F}}\mathbf{v} = -ar_0\dot{\theta}e^{-a\theta}\mathbf{e}_r + r_0\dot{\theta}e^{-a\theta}\mathbf{e}_\theta \tag{2.164}$$

Simplifying Eq. (2.161), we obtain $^{\mathcal{F}}\mathbf{v}$ as

$$^{\mathcal{F}}\mathbf{v} = r_0\dot{\theta}e^{-a\theta}\left[-a\mathbf{e}_r + \mathbf{e}_\theta\right] \tag{2.165}$$

The tangent vector in reference frame $\mathcal{F}$ is then given as

$$\mathbf{e}_t = \frac{^{\mathcal{F}}\mathbf{v}}{\|^{\mathcal{F}}\mathbf{v}\|} = \frac{^{\mathcal{F}}\mathbf{v}}{\|^{\mathcal{F}}v\|} \tag{2.166}$$

where $^{\mathcal{F}}v$ is the speed of the particle in reference frame $\mathcal{F}$. Now the speed of the particle in reference frame $\mathcal{F}$ is given as

$$^{\mathcal{F}}v = \|^{\mathcal{F}}\mathbf{v}\| = r_0\dot{\theta}e^{-a\theta}\sqrt{1+a^2} \tag{2.167}$$

Dividing $^{\mathcal{F}}\mathbf{v}$ in Eq. (2.165) by $^{\mathcal{F}}v$ in Eq. (2.167), we obtain the tangent vector in reference frame $\mathcal{F}$ as

$$\mathbf{e}_t = \frac{-a\mathbf{e}_r + \mathbf{e}_\theta}{\sqrt{1+a^2}} \tag{2.168}$$

Next, the principle unit normal vector is obtained as

$$\mathbf{e}_n = \frac{^{\mathcal{F}}d\mathbf{e}_t/dt}{\|^{\mathcal{F}}d\mathbf{e}_t/dt\|} \tag{2.169}$$

Now we have from the rate of change transport theorem that

$$\frac{{}^{\mathcal{F}}d\mathbf{e}_t}{dt} = \frac{{}^{\mathcal{A}}d\mathbf{e}_t}{dt} + {}^{\mathcal{F}}\boldsymbol{\omega}^{\mathcal{A}} \times \mathbf{e}_t \tag{2.170}$$

where

$$\frac{{}^{\mathcal{F}}d\mathbf{e}_t}{dt} = \mathbf{0} \tag{2.171}$$

$${}^{\mathcal{F}}\boldsymbol{\omega}^{\mathcal{A}} \times \mathbf{e}_t = \dot{\theta}\mathbf{e}_z \times \frac{-a\mathbf{e}_r + \mathbf{e}_\theta}{\sqrt{1+a^2}} = -\dot{\theta}\frac{\mathbf{e}_r + a\mathbf{e}_\theta}{\sqrt{1+a^2}} \tag{2.172}$$

Adding Eq. (2.171) and Eq. (2.172), we obtain

$$\frac{{}^{\mathcal{F}}d\mathbf{e}_t}{dt} = -\dot{\theta}\frac{\mathbf{e}_r + a\mathbf{e}_\theta}{\sqrt{1+a^2}} \tag{2.173}$$

Consequently,

$$\left\| \frac{{}^{\mathcal{F}}d\mathbf{e}_t}{dt} \right\| = \dot{\theta} \tag{2.174}$$

Dividing ${}^{\mathcal{F}}d\mathbf{e}_t/dt$ in Eq. (2.173) by $\|{}^{\mathcal{F}}d\mathbf{e}_t/dt\|$ in Eq. (2.174), we obtain $\mathbf{e}_n$ as

$$\mathbf{e}_n = -\frac{\mathbf{e}_r + a\mathbf{e}_\theta}{\sqrt{1+a^2}} \tag{2.175}$$

Finally, the bi-normal vector is obtained as

$$\mathbf{e}_b = \mathbf{e}_t \times \mathbf{e}_n = \frac{-a\mathbf{e}_r + \mathbf{e}_\theta}{\sqrt{1+a^2}} \times -\frac{\mathbf{e}_r + a\mathbf{e}_\theta}{\sqrt{1+a^2}} = \mathbf{E}_z \tag{2.176}$$

(b) Curvature

The curvature of the trajectory in reference frame $\mathcal{F}$ is then obtained as

$$\kappa = \frac{\|{}^{\mathcal{F}}d\mathbf{e}_t/dt\|}{{}^{\mathcal{F}}v} \tag{2.177}$$

Substituting $\|{}^{\mathcal{F}}d\mathbf{e}_t/dt\|$ from Eq. (2.174) and ${}^{\mathcal{F}}v$ from Eq. (2.167), we obtain κ as

$$\kappa = \frac{1}{r_0 e^{-a\theta}\sqrt{1+a^2}} \tag{2.178}$$

(c) Velocity and Acceleration

The velocity of the particle in reference frame $\mathcal{F}$ can be expressed in the intrinsic basis as

$${}^{\mathcal{F}}\mathbf{v} = {}^{\mathcal{F}}v\,\mathbf{e}_t \tag{2.179}$$

Using the expression for ${}^{\mathcal{F}}v$ from Eq. (2.167), we obtain

$$ {}^{\mathcal{F}}\mathbf{v} = r_0\dot{\theta}e^{-a\theta}\sqrt{1+a^2}\mathbf{e}_t \tag{2.180} $$

Next, the acceleration of the particle in reference frame $\mathcal{F}$ can be expressed in terms of the intrinsic basis as

$$ {}^{\mathcal{F}}\mathbf{a} = \frac{d}{dt}\left({}^{\mathcal{F}}v\right)\mathbf{e}_t + \kappa\left({}^{\mathcal{F}}v\right)^2\mathbf{e}_n \tag{2.181} $$

Now we have that

$$ \frac{d}{dt}\left({}^{\mathcal{F}}v\right) = r_0(\ddot{\theta} - a\dot{\theta}^2)e^{-a\theta}\sqrt{1+a^2} \tag{2.182} $$

Furthermore,

$$ \kappa\left({}^{\mathcal{F}}v\right)^2 = \frac{1}{r_0e^{-a\theta}\sqrt{1+a^2}}r_0^2\dot{\theta}^2e^{-2a\theta}(1+a^2) = r_0\dot{\theta}^2e^{-a\theta}\sqrt{1+a^2} \tag{2.183} $$

The acceleration in reference frame $\mathcal{F}$ is then given as

$$ {}^{\mathcal{F}}\mathbf{a} = r_0e^{-a\theta}\sqrt{1+a^2}(\ddot{\theta} - a\dot{\theta}^2)\mathbf{e}_t + r_0\dot{\theta}^2e^{-a\theta}\sqrt{1+a^2}\mathbf{e}_n \tag{2.184} $$

Question 2–15

A circular disk of radius R is attached to a rotating shaft of length L as shown in Fig. P2-15. The shaft rotates about the vertical direction with a constant angular velocity $\boldsymbol{\Omega}$ relative to the ground. The disk, in turn, rotates about its center about an axis orthogonal to the shaft. Knowing that the angle θ describes the position of a point P located on the edge of the disk relative to the center of the disk, determine the following quantities as viewed by an observer fixed to the ground: (a) the angular velocity of the disk and (b) the velocity and acceleration of point P.

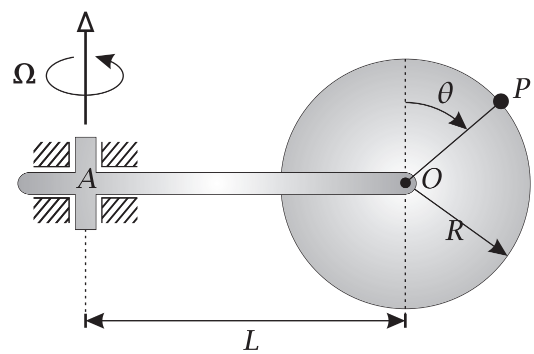

Figure P2-15

Solution to Question 2–15

First, let $\mathcal{F}$ be a reference frame fixed to the ground. Then, we choose the following coordinate system fixed in reference frame $\mathcal{F}$:

	Origin at Point A	
$\mathbf{E}_x$	=	Along $\boldsymbol{\Omega}$
$\mathbf{E}_y$	=	Along AO at $t = 0$
$\mathbf{E}_z$	=	$\mathbf{E}_x \times \mathbf{E}_y$

Next, let $\mathcal{A}$ be a reference frame fixed to the horizontal shaft. Then, we choose the following coordinate system fixed in reference frame $\mathcal{F}$:

	Origin at Point A	
$\mathbf{e}_x$	=	Along $\boldsymbol{\Omega}$
$\mathbf{e}_y$	=	Along AO
$\mathbf{e}_z$	=	$\mathbf{e}_x \times \mathbf{e}_y$

Lastly, let $\mathcal{B}$ be a reference frame fixed to the disk. Then, choose the following coordinate system fixed in reference frame $\mathcal{B}$:

$$
\begin{array}{ccl}
 & \text{Origin at Point } O & \\
\mathbf{e}_r & = & \text{Along } OP \\
\mathbf{e}_z & = & \text{Same as Reference Frame } \mathcal{A} \\
\mathbf{e}_\theta & = & \mathbf{e}_z \times \mathbf{e}_r
\end{array}
$$

The geometry of the bases $\{\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z\}$ and $\{\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_z\}$ is shown in Fig. (2.185). In particular, using Fig. (2.185), we have that

$$
\begin{array}{rcl}
\mathbf{e}_x & = & \cos\theta\,\mathbf{e}_r - \sin\theta\,\mathbf{e}_\theta \\
\mathbf{e}_y & = & \sin\theta\,\mathbf{e}_r + \cos\theta\,\mathbf{e}_\theta
\end{array} \tag{2.185}
$$

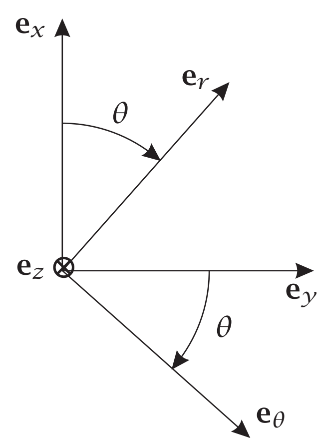

Figure 2-3 Relationship Between Basis $\{\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z\}$ and $\{\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_z\}$ for Question 2–15

Now, since the shaft rotates with angular velocity $\boldsymbol{\Omega}$ about the $\mathbf{e}_y$-direction relative to the ground, the angular velocity of reference frame $\mathcal{A}$ in reference frame $\mathcal{F}$ is given as

$$
{}^{\mathcal{F}}\boldsymbol{\omega}^{\mathcal{A}} = \boldsymbol{\Omega} = \Omega\mathbf{e}_x \tag{2.186}
$$

Next, since the disk rotates with angular rate $\dot{\theta}$ relative to the shaft in the $\mathbf{e}_z$-direction, the angular velocity of reference frame $\mathcal{B}$ in reference frame $\mathcal{A}$ is given as

$$
{}^{\mathcal{A}}\boldsymbol{\omega}^{\mathcal{B}} = \dot{\theta}\mathbf{e}_z \tag{2.187}
$$

The angular velocity of reference frame $\mathcal{B}$ in reference frame $\mathcal{F}$ is then obtained using the theorem of angular velocity addition as

$$
{}^{\mathcal{F}}\boldsymbol{\omega}^{\mathcal{B}} = {}^{\mathcal{F}}\boldsymbol{\omega}^{\mathcal{A}} + {}^{\mathcal{A}}\boldsymbol{\omega}^{\mathcal{B}} = \Omega\mathbf{e}_x + \dot{\theta}\mathbf{e}_z \tag{2.188}
$$

Then, using the relationship between $\{\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z\}$ and $\{\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_z\}$ from Eq. (2.185), we obtain ${}^{\mathcal{F}}\boldsymbol{\omega}^{\mathcal{B}}$ in terms of the basis $\{\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_z\}$ as

$$ {}^{\mathcal{F}}\boldsymbol{\omega}^{\mathcal{B}} = \Omega(\cos\theta\,\mathbf{e}_r - \sin\theta\,\mathbf{e}_\theta) + \dot{\theta}\mathbf{e}_z = \Omega\cos\theta\,\mathbf{e}_r - \Omega\sin\theta\,\mathbf{e}_\theta + \dot{\theta}\mathbf{e}_z \tag{2.189} $$

Next, the position of point P is given as

$$ \mathbf{r}_P = \mathbf{r}_O + \mathbf{r}_{P/O} \tag{2.190} $$

Now, in terms of the basis $\{\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z\}$, the position of point O is given as

$$ \mathbf{r}_O = L\mathbf{e}_y \tag{2.191} $$

Also, in terms of the basis $\{\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_z\}$ we have that

$$ \mathbf{r}_{P/O} = R\mathbf{e}_r \tag{2.192} $$

Consequently,

$$ \mathbf{r}_P = L\mathbf{e}_y + R\mathbf{e}_r \tag{2.193} $$

The velocity of point P in reference frame $\mathcal{F}$ is then given as

$$ {}^{\mathcal{F}}\mathbf{v}_P = \frac{{}^{\mathcal{F}}d}{dt}(\mathbf{r}_P) = \frac{{}^{\mathcal{F}}d}{dt}(\mathbf{r}_O) + \frac{{}^{\mathcal{F}}d}{dt}(\mathbf{r}_{O/P}) = {}^{\mathcal{F}}\mathbf{v}_O + {}^{\mathcal{F}}\mathbf{v}_{P/O} \tag{2.194} $$

First, since $\mathbf{r}_O$ is expressed in terms of the basis $\{\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z\}$ and $\{\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z\}$ is fixed in reference frame $\mathcal{A}$, we can apply the rate of change transport theorem to $\mathbf{r}_O$ between reference frames $\mathcal{A}$ and $\mathcal{F}$ as

$$ {}^{\mathcal{F}}\mathbf{v}_O = \frac{{}^{\mathcal{F}}d}{dt}(\mathbf{r}_O) = \frac{{}^{\mathcal{A}}d}{dt}(\mathbf{r}_O) + {}^{\mathcal{F}}\boldsymbol{\omega}^{\mathcal{A}} \times \mathbf{r}_O \tag{2.195} $$

Now we have that

$$ \begin{aligned} \frac{{}^{\mathcal{A}}d}{dt}(\mathbf{r}_O) &= \mathbf{0} && (2.196) \\ {}^{\mathcal{F}}\boldsymbol{\omega}^{\mathcal{A}} \times \mathbf{r}_O &= \Omega\mathbf{e}_x \times L\mathbf{e}_y = L\Omega\mathbf{e}_z && (2.197) \end{aligned} $$

Adding Eq. (2.196) and Eq. (2.197), we obtain

$$ {}^{\mathcal{F}}\mathbf{v}_O = L\Omega\mathbf{e}_z \tag{2.198} $$

Next, since $\mathbf{r}_{P/O}$ is expressed in terms of the basis $\{\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_z\}$ and $\{\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_z\}$ is fixed in reference frame $\mathcal{B}$, we can apply the rate of change transport theorem to $\mathbf{r}_{P/O}$ between reference frames $\mathcal{B}$ and $\mathcal{F}$ as

$$ {}^{\mathcal{F}}\mathbf{v}_{P/O} = \frac{{}^{\mathcal{F}}d}{dt}(\mathbf{r}_{P/O}) = \frac{{}^{\mathcal{B}}d}{dt}(\mathbf{r}_{P/O}) + {}^{\mathcal{F}}\boldsymbol{\omega}^{\mathcal{B}} \times \mathbf{r}_{P/O} \tag{2.199} $$

Now we have that

$$
\begin{aligned}
\frac{{}^{\mathcal{B}}d}{dt}\left(\mathbf{r}_{P/O}\right) &= \mathbf{0} && (2.200)\\
{}^{\mathcal{F}}\boldsymbol{\omega}^{\mathcal{B}} \times \mathbf{r}_{P/O} &= (\Omega\cos\theta\,\mathbf{e}_r - \Omega\sin\theta\,\mathbf{e}_\theta + \dot{\theta}\mathbf{e}_z) \times R\mathbf{e}_r \\
&= R\dot{\theta}\mathbf{e}_\theta + R\Omega\sin\theta\,\mathbf{e}_z && (2.201)
\end{aligned}
$$

Adding Eq. (2.200) and Eq. (2.201), we obtain

$$
{}^{\mathcal{F}}\mathbf{v}_{P/O} = R\dot{\theta}\mathbf{e}_\theta + R\Omega\sin\theta\,\mathbf{e}_z \qquad (2.202)
$$

The velocity of point P in reference frame $\mathcal{F}$ is then obtained by adding Eq. (2.198) and Eq. (2.202) as

$$
{}^{\mathcal{F}}\mathbf{v}_P = {}^{\mathcal{F}}\mathbf{v}_O + {}^{\mathcal{F}}\mathbf{v}_{P/O} = L\Omega\mathbf{e}_z + R\dot{\theta}\mathbf{e}_\theta + R\Omega\sin\theta\,\mathbf{e}_z \qquad (2.203)
$$

Simplifying Eq. (2.203), we obtain

$$
{}^{\mathcal{F}}\mathbf{v}_P = R\dot{\theta}\mathbf{e}_\theta + (L + R\sin\theta)\Omega\mathbf{e}_z \qquad (2.204)
$$

The acceleration of point P in reference frame $\mathcal{F}$ is obtained in the same manner as was used to obtain the velocity in reference frame $\mathcal{F}$. First, we have from Eq. (2.203) that

$$
{}^{\mathcal{F}}\mathbf{v}_P = {}^{\mathcal{F}}\mathbf{v}_O + {}^{\mathcal{F}}\mathbf{v}_{P/O} \qquad (2.205)
$$

Now, since ${}^{\mathcal{F}}\mathbf{v}_O$ is expressed in the basis $\{\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z\}$ and $\{\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z\}$ is fixed in reference frame $\mathcal{A}$, the acceleration of point O in reference frame $\mathcal{F}$ can be obtained by applying the rate of change transport theorem to ${}^{\mathcal{F}}\mathbf{v}_O$ between reference frames $\mathcal{A}$ and $\mathcal{F}$ as

$$
{}^{\mathcal{F}}\mathbf{a}_O = \frac{{}^{\mathcal{F}}d}{dt}\left({}^{\mathcal{F}}\mathbf{v}_O\right) = \frac{{}^{\mathcal{A}}d}{dt}\left({}^{\mathcal{F}}\mathbf{v}_O\right) + {}^{\mathcal{F}}\boldsymbol{\omega}^{\mathcal{A}} \times {}^{\mathcal{F}}\mathbf{v}_O \qquad (2.206)
$$

Now we have that

$$
\begin{aligned}
\frac{{}^{\mathcal{A}}d}{dt}\left({}^{\mathcal{F}}\mathbf{v}_O\right) &= \mathbf{0} && (2.207)\\
{}^{\mathcal{F}}\boldsymbol{\omega}^{\mathcal{A}} \times {}^{\mathcal{F}}\mathbf{v}_O &= \Omega\mathbf{e}_x \times L\Omega\mathbf{e}_z = -L\Omega^2\mathbf{e}_y && (2.208)
\end{aligned}
$$

Then, adding Eq. (2.207) and Eq. (2.208), we obtain ${}^{\mathcal{F}}\mathbf{a}_O$ as

$$
{}^{\mathcal{F}}\mathbf{a}_O = -L\Omega^2\mathbf{e}_y \qquad (2.209)
$$

Next, since ${}^{\mathcal{F}}\mathbf{v}_{P/O}$ is expressed in terms of the basis $\{\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_z\}$, the acceleration of point P relative to point O in reference frame $\mathcal{F}$ is obtained by applying the rate of change transport theorem to ${}^{\mathcal{F}}\mathbf{v}_{P/O}$ between reference frames $\mathcal{B}$ and $\mathcal{F}$ as

$$
{}^{\mathcal{F}}\mathbf{a}_{P/O} = \frac{{}^{\mathcal{F}}d}{dt}\left({}^{\mathcal{F}}\mathbf{v}_{P/O}\right) = \frac{{}^{\mathcal{B}}d}{dt}\left({}^{\mathcal{F}}\mathbf{v}_{P/O}\right) + {}^{\mathcal{F}}\boldsymbol{\omega}^{\mathcal{B}} \times {}^{\mathcal{F}}\mathbf{v}_{P/O} \qquad (2.210)
$$

$$
\begin{aligned}
\frac{{}^{\mathcal{B}}d}{dt}\left({}^{\mathcal{F}}\mathbf{v}_{P/O}\right) &= R\ddot{\theta}\mathbf{e}_\theta + R\Omega\dot{\theta}\cos\theta\,\mathbf{e}_z && (2.211)\\
{}^{\mathcal{F}}\boldsymbol{\omega}^{\mathcal{B}}\times{}^{\mathcal{F}}\mathbf{v}_O &= (\Omega\cos\theta\,\mathbf{e}_r - \Omega\sin\theta\,\mathbf{e}_\theta + \dot{\theta}\mathbf{e}_z)\times(R\dot{\theta}\mathbf{e}_\theta + R\Omega\sin\theta\,\mathbf{e}_z)\\
&= R\Omega\dot{\theta}\cos\theta\,\mathbf{e}_z - R\Omega^2\cos\theta\,\sin\theta\,\mathbf{e}_\theta\\
&\quad -R\Omega^2\sin^2\theta\mathbf{e}_r - R\dot{\theta}^2\mathbf{e}_r && (2.212)
\end{aligned}
$$

Simplifying Eq. (2.212), we obtain

$$
{}^{\mathcal{F}}\boldsymbol{\omega}^{\mathcal{B}}\times{}^{\mathcal{F}}\mathbf{v}_O = -(R\dot{\theta}^2 + R\Omega^2\sin^2\theta)\mathbf{e}_r - R\Omega^2\cos\theta\,\sin\theta\,\mathbf{e}_\theta + R\Omega\dot{\theta}\cos\theta\,\mathbf{e}_z \quad (2.213)
$$

Then, adding Eq. (2.211) and Eq. (2.213), we obtain ${}^{\mathcal{F}}\mathbf{a}_{P/O}$ as

$$
{}^{\mathcal{F}}\mathbf{a}_{P/O} = -(R\dot{\theta}^2 + R\Omega^2\sin^2\theta)\mathbf{e}_r + (R\ddot{\theta} - R\Omega^2\cos\theta\,\sin\theta)\mathbf{e}_\theta + 2R\Omega\dot{\theta}\cos\theta\,\mathbf{e}_z \quad (2.214)
$$

Finally, adding Eq. (2.209) and Eq. (2.214), we obtain ${}^{\mathcal{F}}\mathbf{a}_P$ as

$$
{}^{\mathcal{F}}\mathbf{a}_P = -L\Omega^2\mathbf{e}_y - (R\dot{\theta}^2 + R\Omega^2\sin^2\theta)\mathbf{e}_r + (R\ddot{\theta} - R\Omega^2\cos\theta\,\sin\theta)\mathbf{e}_\theta + 2R\Omega\dot{\theta}\cos\theta\,\mathbf{e}_z \quad (2.215)
$$

Now it is seen from Eq. (2.215) that some of the terms in ${}^{\mathcal{F}}\mathbf{a}_P$ are expressed in the basis $\{\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z\}$ while other terms are expressed in the basis $\{\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_z\}$. However, using Eq. (2.185), we can obtain an expression for ${}^{\mathcal{F}}\mathbf{a}_P$ in terms of a single basis. Now, while it is possible to write ${}^{\mathcal{F}}\mathbf{a}_P$ in terms of the basis $\{\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z\}$, it is preferable (and simpler) to write both quantities in terms of the basis $\{\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_z\}$. First, substituting the expression for $\mathbf{e}_y$ from Eq. (2.185) into Eq. (2.215), we obtain ${}^{\mathcal{F}}\mathbf{a}_P$ in terms of $\{\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_z\}$ as

$$
\begin{aligned}
{}^{\mathcal{F}}\mathbf{a}_P = &-L\Omega^2(\sin\theta\,\mathbf{e}_r + \cos\theta\,\mathbf{e}_\theta) - (R\dot{\theta}^2 + R\Omega^2\sin^2\theta)\mathbf{e}_r\\
&+ (R\ddot{\theta} - R\Omega^2\cos\theta\,\sin\theta)\mathbf{e}_\theta + 2R\Omega\dot{\theta}\cos\theta\,\mathbf{e}_z
\end{aligned} \quad (2.216)
$$

Simplifying Eq. (2.216) gives

$$
\begin{aligned}
{}^{\mathcal{F}}\mathbf{a}_P = &-(L\Omega^2\sin\theta + R\dot{\theta}^2 + R\Omega^2\sin^2\theta)\mathbf{e}_r\\
&+ (R\ddot{\theta} - L\Omega^2\cos\theta - R\Omega^2\cos\theta\,\sin\theta)\mathbf{e}_\theta\\
&+ 2R\Omega\dot{\theta}\cos\theta\,\mathbf{e}_z
\end{aligned} \quad (2.217)
$$

Question 2–16

A disk of radius R rotates freely about its center at a point located on the end of an arm of length L as shown in Fig. P2-16. The arm itself pivots freely at its other end at point O to a vertical shaft. Finally, the shaft rotates with constant angular velocity $\mathbf{\Omega}$ relative to the ground. Knowing that ϕ describes the location of a point P on the edge of the disk relative to the direction OQ and that θ is formed by the arm with the downward direction, determine the following quantities as viewed by an observer fixed to the ground: (a) the angular velocity of the disk and (b) the velocity and acceleration of point P.

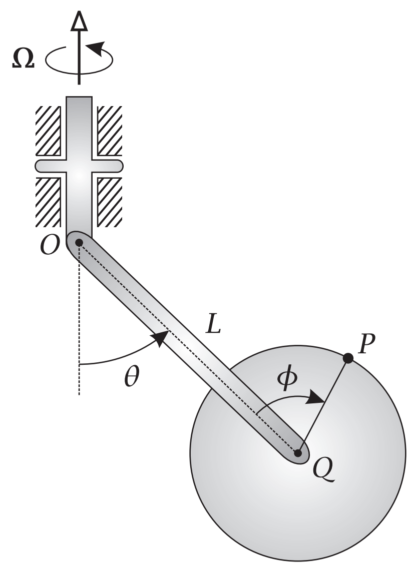

Figure P2-16

Solution to Question 2–16

Let $\mathcal{F}$ be a reference frame fixed to the ground. Then choose the following coordinate system fixed in reference frame $\mathcal{F}$:

	Origin at O	
$\mathbf{E}_x$	=	Along OQ when $\theta = 0$
$\mathbf{E}_z$	=	Orthogonal to plane of shaft and arm and out of page at $t = 0$
$\mathbf{E}_y$	=	$\mathbf{E}_z \times \mathbf{E}_x$

Next, let $\mathcal{A}$ be a reference frame fixed to the vertical shaft. Then choose the following coordinate system fixed in reference frame $\mathcal{A}$:

$$\begin{array}{ccl} & \text{Origin at } O & \\ \mathbf{e}_x & = & \mathbf{E}_x \\ \mathbf{e}_z & = & \text{Orthogonal to plane of shaft and arm} \\ \mathbf{e}_y & = & \mathbf{e}_z \times \mathbf{e}_x \end{array}$$

We note that $\mathbf{e}_z$ and $\mathbf{E}_z$ are equal when $t = 0$. Next, let $\mathcal{B}$ be a reference frame fixed to the rod OQ. Then choose the following coordinate system fixed in reference frame $\mathcal{B}$:

$$\begin{array}{ccl} & \text{Origin at } O & \\ \mathbf{u}_x & = & \text{Along } OQ \\ \mathbf{u}_z & = & \text{Orthogonal to plane of shaft, arm, and disk} \\ \mathbf{u}_y & = & \mathbf{u}_z \times \mathbf{u}_x \end{array}$$

Finally, let $\mathcal{D}$ be a reference frame fixed to the disk. Then choose the following coordinate system fixed in reference frame $\mathcal{D}$:

$$\begin{array}{ccl} & \text{Origin at } O & \\ \mathbf{i}_r & = & \text{Along } OQ \\ \mathbf{i}_z & = & -\mathbf{e}_z \\ \mathbf{i}_\phi & = & \mathbf{u}_z \times \mathbf{u}_r \end{array}$$

The geometry of the bases $\{\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z\}$, $\{\mathbf{u}_x, \mathbf{u}_y, \mathbf{u}_z\}$, and $\{\mathbf{i}_r, \mathbf{i}_\phi, \mathbf{i}_z\}$ is shown in Fig. 2-4.

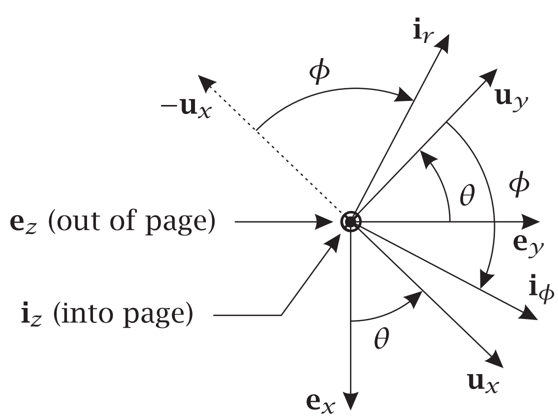

Figure 2-4 Geometry of bases $\{\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z\}$, $\{\mathbf{u}_x, \mathbf{u}_y, \mathbf{u}_z\}$, and $\{\mathbf{i}_r, \mathbf{i}_\phi, \mathbf{i}_z\}$ for Question 2–16.

The angular velocity of reference frame $\mathcal{A}$ in reference frame $\mathcal{F}$ is then given as

$$^{\mathcal{F}}\boldsymbol{\omega}^{\mathcal{A}} = -\boldsymbol{\Omega} = -\Omega\mathbf{e}_x \tag{2.218}$$

where the negative sign arises from the fact that the positive sense of $\boldsymbol{\Omega}$ is vertically *upward* while the direction $\mathbf{E}_x$ is *downward*. Next, the angular velocity of reference frame $\mathcal{B}$ in reference frame $\mathcal{A}$ is given as

$$ {}^{\mathcal{A}}\boldsymbol{\omega}^{\mathcal{B}} = \dot{\theta}\mathbf{u}_z \tag{2.219} $$

Next, the angular velocity of reference frame $\mathcal{D}$ in reference frame $\mathcal{B}$ is given as

$$ {}^{\mathcal{B}}\boldsymbol{\omega}^{\mathcal{D}} = \dot{\phi}\mathbf{i}_z = -\phi\mathbf{u}_z \tag{2.220} $$

The angular velocity of the disk as viewed by an observer fixed to the ground is then obtained from the angular velocity addition theorem as

$$ {}^{\mathcal{F}}\boldsymbol{\omega}^{\mathcal{D}} = {}^{\mathcal{F}}\boldsymbol{\omega}^{\mathcal{A}} + {}^{\mathcal{A}}\boldsymbol{\omega}^{\mathcal{B}} + {}^{\mathcal{B}}\boldsymbol{\omega}^{\mathcal{D}} = -\Omega\mathbf{e}_x + \dot{\theta}\mathbf{u}_z - \dot{\phi}\mathbf{u}_z \tag{2.221} $$

Now from the geometry of the bases, it is seen that

$$ \mathbf{e}_x = \cos\theta\,\mathbf{u}_x - \sin\theta\,\mathbf{u}_y \tag{2.222} $$

which implies that

$$ \begin{aligned} {}^{\mathcal{F}}\boldsymbol{\omega}^{\mathcal{D}} &= -\Omega(\cos\theta\,\mathbf{u}_x - \sin\theta\,\mathbf{u}_y) + (\dot{\theta} - \dot{\phi})\mathbf{u}_z \\ &= -\Omega\cos\theta\,\mathbf{u}_x + \Omega\sin\theta\,\mathbf{u}_y + (\dot{\theta} - \dot{\phi})\mathbf{u}_z \end{aligned} \tag{2.223} $$

Now because point P (i.e., the point for which we want the velocity) is fixed to the disk, it is helpful to obtain an expression for ${}^{\mathcal{F}}\boldsymbol{\omega}^{\mathcal{D}}$ in terms of the basis $\{\mathbf{i}_r, \mathbf{i}_\phi, \mathbf{i}_z\}$. In order to obtain such an expression, it is first important to see from Fig. 2-4 that

$$ -\mathbf{u}_x = \cos\phi\,\mathbf{i}_r - \sin\phi\,\mathbf{i}_\phi \Longrightarrow \mathbf{u}_x = -\cos\phi\,\mathbf{i}_r + \sin\phi\,\mathbf{i}_\phi \tag{2.224} $$

$$ \mathbf{u}_y = \sin\phi\,\mathbf{i}_r + \cos\phi\,\mathbf{i}_\phi \tag{2.225} $$

where it is observed that diagramatically it is first easier to determine $-\mathbf{u}_x$ in terms of $\mathbf{i}_r$ and $\mathbf{i}_\phi$ and then take the negative sign of the result. Consequently,

$$ \begin{aligned} {}^{\mathcal{F}}\boldsymbol{\omega}^{\mathcal{D}} &= -\Omega\cos\theta\,(-\cos\phi\,\mathbf{i}_r + \sin\phi\,\mathbf{i}_\phi) + \Omega\sin\theta\,(\sin\phi\,\mathbf{i}_r + \cos\phi\,\mathbf{i}_\phi) + (\dot{\theta} - \dot{\phi})\mathbf{u}_z \\ &= \Omega(\cos\theta\,\cos\phi + \sin\theta\,\sin\phi)\mathbf{i}_r + \Omega(\cos\theta\,\sin\phi - \sin\theta\,\cos\phi)\mathbf{i}_\phi + (\dot{\theta} - \dot{\phi})\mathbf{i}_z \\ &= \Omega\cos(\theta - \phi)\mathbf{i}_r + \Omega\sin(\theta - \phi)\mathbf{i}_\phi + (\dot{\theta} - \dot{\phi})\mathbf{i}_z \end{aligned} \tag{2.226} $$

where we have used the two trigonometric identities

$$ \cos\theta\,\cos\phi + \sin\theta\,\sin\phi = \cos(\theta - \phi) \tag{2.227} $$

$$ \cos\theta\,\sin\phi - \sin\theta\,\cos\phi = \sin(\theta - \phi) \tag{2.228} $$

Now we know that the position of point P is given as

$$\mathbf{r}_P = \mathbf{r}_Q + \mathbf{r}_{P/Q} \tag{2.229}$$

where

$$\begin{aligned}\mathbf{r}_Q &= L\mathbf{u}_x &(2.230)\\ \mathbf{r}_{P/Q} &= R\mathbf{i}_r &(2.231)\end{aligned}$$

Because the basis $\{\mathbf{u}_x, \mathbf{u}_y, \mathbf{u}_z\}$ is fixed in reference frame $\mathcal{B}$, we can apply the transport theorem to $\mathbf{r}_Q$ between reference frames $\mathcal{B}$ and $\mathcal{F}$ as

$${}^{\mathcal{F}}\mathbf{v}_Q = \frac{{}^{\mathcal{F}}d\mathbf{r}_Q}{dt} = \frac{{}^{\mathcal{B}}d\mathbf{r}_Q}{dt} + {}^{\mathcal{F}}\boldsymbol{\omega}^{\mathcal{B}} \times \mathbf{r}_Q \tag{2.232}$$

Now we have

$${}^{\mathcal{F}}\boldsymbol{\omega}^{\mathcal{B}} = {}^{\mathcal{F}}\boldsymbol{\omega}^{\mathcal{A}} + {}^{\mathcal{A}}\boldsymbol{\omega}^{\mathcal{B}} = -\Omega\mathbf{e}_x + \dot{\theta}\mathbf{u}_z = -\Omega\cos\theta\,\mathbf{u}_x + \Omega\sin\theta\,\mathbf{u}_y + \dot{\theta}\mathbf{u}_z \tag{2.233}$$

Furthermore,

$$\begin{aligned}\frac{{}^{\mathcal{B}}d\mathbf{r}_Q}{dt} &= \mathbf{0} &(2.234)\\ {}^{\mathcal{F}}\boldsymbol{\omega}^{\mathcal{B}} \times \mathbf{r}_Q &= (-\Omega\cos\theta\,\mathbf{u}_x + \Omega\sin\theta\,\mathbf{u}_y + \dot{\theta}\mathbf{u}_z) \times L\mathbf{u}_x &(2.235)\end{aligned}$$

which gives

$${}^{\mathcal{F}}\boldsymbol{\omega}^{\mathcal{B}} \times \mathbf{r}_Q = (-\Omega\cos\theta\,\mathbf{u}_x + \Omega\sin\theta\,\mathbf{u}_y + \dot{\theta}\mathbf{u}_z) \times L\mathbf{u}_x = L\dot{\theta}\mathbf{u}_y - L\Omega\sin\theta\,\mathbf{u}_z \tag{2.236}$$

Therefore,

$${}^{\mathcal{F}}\mathbf{v}_Q = L\dot{\theta}\mathbf{u}_y - L\Omega\sin\theta\,\mathbf{u}_z \tag{2.237}$$

Next, because $\mathbf{r}_{P/Q}$ is fixed in reference frame $\mathcal{D}$, we can apply the transport theorem to $\mathbf{r}_{P/Q}$ between reference frames $\mathcal{D}$ and $\mathcal{F}$ as

$${}^{\mathcal{F}}\mathbf{v}_{P/Q} = \frac{{}^{\mathcal{F}}d}{dt}(\mathbf{r}_{P/Q}) = \frac{{}^{\mathcal{D}}d}{dt}(\mathbf{r}_{P/Q}) + {}^{\mathcal{F}}\boldsymbol{\omega}^{\mathcal{D}} \times \mathbf{r}_{P/Q} \tag{2.238}$$

Now we have

$$\begin{aligned}\frac{{}^{\mathcal{D}}d}{dt}(\mathbf{r}_{P/Q}) &= \mathbf{0} &(2.239)\\ {}^{\mathcal{F}}\boldsymbol{\omega}^{\mathcal{B}} \times \mathbf{r}_{P/Q} &= [\Omega\cos(\theta-\phi)\mathbf{i}_r + \Omega\sin(\theta-\phi)\mathbf{i}_\phi + (\dot{\theta}-\dot{\phi})\mathbf{i}_z] \\ &\quad \times R\mathbf{i}_r &(2.240)\end{aligned}$$

which implies that

$$\begin{aligned}{}^{\mathcal{F}}\boldsymbol{\omega}^{\mathcal{B}} \times \mathbf{r}_{P/Q} &= [\Omega\cos(\theta-\phi)\mathbf{i}_r + \Omega\sin(\theta-\phi)\mathbf{i}_\phi + (\dot{\theta}-\dot{\phi})\mathbf{i}_z] \times R\mathbf{i}_r \\ &= R(\dot{\theta}-\dot{\phi})\mathbf{i}_\phi - R\Omega\sin(\theta-\phi)\mathbf{i}_z\end{aligned} \tag{2.241}$$

Therefore,

$$^{\mathcal{F}}\mathbf{v}_{P/Q} = R(\dot{\theta} - \dot{\phi})\mathbf{i}_{\phi} - R\Omega\sin(\theta - \phi)\mathbf{i}_z \tag{2.242}$$

The velocity of point P in reference frame $\mathcal{F}$ is then obtained by adding Eqs. (2.237) and (2.242) as

$$^{\mathcal{F}}\mathbf{v}_P = L\dot{\theta}\mathbf{u}_y - L\Omega\sin\theta\mathbf{u}_z + +R(\dot{\theta} - \dot{\phi})\mathbf{i}_{\phi} - R\Omega\sin(\theta - \phi)\mathbf{i}_z \tag{2.243}$$

It is noted that this last expression can be converted to an expression in terms of a single basis using the relationships between the bases as given in Fig. 2-4.

Question 2–17

A particle slides along a track in the form of a spiral as shown in Fig. P2-17. The equation for the spiral is

$$r = a\theta$$

where a is a constant and θ is the angle measured from the horizontal. Determine (a) expressions for the intrinsic basis vectors $\mathbf{e}_t$, $\mathbf{e}_n$, and $\mathbf{e}_b$ in terms any other basis of your choosing, (b) determine the curvature of the trajectory as a function of the angle θ, and (c) determine the velocity and acceleration of the collar as viewed by an observer fixed to the track.

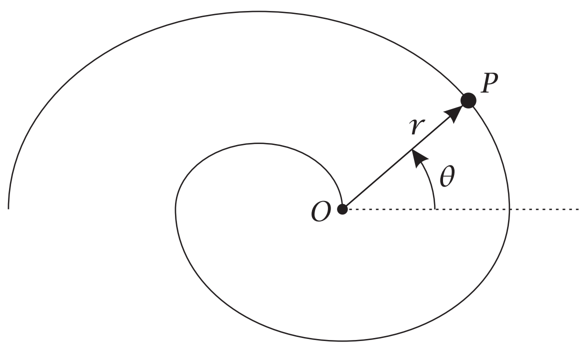

Figure P2-17

Solution to Question 2–17

First, let $\mathcal{F}$ be a reference frame fixed to the spiral. Then, choose the following coordinate system fixed in reference frame $\mathcal{F}$:

	Origin at O	
$\mathbf{E}_x$	=	To the Right
$\mathbf{E}_z$	=	Out of Page
$\mathbf{E}_y$	=	$\mathbf{E}_z \times \mathbf{E}_x$

Next, let$\mathcal{A}$ be a reference frame fixed to direction of OP. Then, choose the following coordinate system fixed in reference frame $\mathcal{A}$:

	Origin at Point O	
$\mathbf{e}_r$	=	Along OP
$\mathbf{E}_z$	=	Out of The Page
$\mathbf{e}_\theta$	=	$\mathbf{E}_z \times \mathbf{e}_r$

Determination of Intrinsic Basis

The position of the particle in terms of the basis $\{\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{E}_z\}$ is given as

$$\mathbf{r} = r\mathbf{e}_r = a\theta\mathbf{e}_r \tag{2.244}$$

Furthermore, the angular velocity of reference frame $\mathcal{A}$ in reference frame $\mathcal{F}$ is given as

$${}^{\mathcal{F}}\boldsymbol{\omega}^{\mathcal{A}} = \dot{\theta}\mathbf{E}_z \tag{2.245}$$

The velocity of the particle in reference frame $\mathcal{F}$ is then obtained using the rate of change transport theorem as

$${}^{\mathcal{F}}\mathbf{v} = \frac{{}^{\mathcal{F}}d\mathbf{r}}{dt} = \frac{{}^{\mathcal{A}}d\mathbf{r}}{dt} + {}^{\mathcal{F}}\boldsymbol{\omega}^{\mathcal{A}} \times \mathbf{r} \tag{2.246}$$

Now we have that

$$\begin{aligned} \frac{{}^{\mathcal{A}}d\mathbf{r}}{dt} &= a\dot{\theta}\mathbf{e}_r \\ {}^{\mathcal{F}}\boldsymbol{\omega}^{\mathcal{A}} \times \mathbf{r} &= \dot{\theta}\mathbf{E}_z \times a\theta\mathbf{e}_r = a\dot{\theta}\theta\mathbf{e}_\theta \end{aligned} \tag{2.247}$$

Adding the two expressions in Eq. (2.247), the velocity of the particle in reference frame $\mathcal{F}$ is obtained as

$${}^{\mathcal{F}}\mathbf{v} = a\dot{\theta}\mathbf{e}_r + a\dot{\theta}\theta\mathbf{e}_\theta \tag{2.248}$$

Eq. (2.248) can be re-written as

$${}^{\mathcal{F}}\mathbf{v} = a\dot{\theta}\left(\mathbf{e}_r + \theta\mathbf{e}_\theta\right) \tag{2.249}$$

The speed of the particle in reference frame $\mathcal{F}$ is then obtained from Eq. (2.249) as

$${}^{\mathcal{F}}v = \|{}^{\mathcal{F}}\mathbf{v}\| = a\dot{\theta}\sqrt{1+\theta^2} = a\dot{\theta}\left(1+\theta^2\right)^{1/2} \tag{2.250}$$

Then, the tangent vector in reference frame $\mathcal{F}$ is obtained as

$$\mathbf{e}_t = \frac{{}^{\mathcal{F}}\mathbf{v}}{{}^{\mathcal{F}}v} \tag{2.251}$$

Then, using ${}^{\mathcal{F}}\mathbf{v}$ from Eq. (2.249) and ${}^{\mathcal{F}}v$ from Eq. (2.250), we obtain the tangent vector in reference frame $\mathcal{F}$ as

$$\mathbf{e}_t = \frac{a\dot{\theta}\left(\mathbf{e}_r + \theta\mathbf{e}_\theta\right)}{a\dot{\theta}\sqrt{1+\theta^2}} = \frac{\mathbf{e}_r + \theta\mathbf{e}_\theta}{\sqrt{1+\theta^2}} = \left(1+\theta^2\right)^{-1/2}\left(\mathbf{e}_r + \theta\mathbf{e}_\theta\right) \tag{2.252}$$

Next, the principle unit normal vector in reference frame $\mathcal{F}$ is obtained as

$$\mathbf{e}_n = \frac{{}^{\mathcal{F}}d\mathbf{e}_t/dt}{\|{}^{\mathcal{F}}d\mathbf{e}_t/dt\|} \tag{2.253}$$

Now, using the rate of change transport theorem, we can compute ${}^{\mathcal{F}}d\mathbf{e}_t/dt$ in reference frame $\mathcal{F}$ as

$$\frac{{}^{\mathcal{F}}d\mathbf{e}_t}{dt} = \frac{{}^{\mathcal{A}}d\mathbf{e}_t}{dt} + {}^{\mathcal{F}}\boldsymbol{\omega}^{\mathcal{A}} \times \mathbf{e}_t \tag{2.254}$$

Using the expression for $\mathbf{e}_t$ from Eq. (2.252), we have that

$$\frac{{}^{\mathcal{A}}d\mathbf{e}_t}{dt} = -\frac{1}{2}\left(1+\theta^2\right)^{-3/2}(2\theta\dot{\theta})(\mathbf{e}_r + \theta\mathbf{e}_\theta) + \left(1+\theta^2\right)^{-1/2}\dot{\theta}\mathbf{e}_\theta \tag{2.255}$$

Eq. (2.255) simplifies to

$$\frac{{}^{\mathcal{A}}d\mathbf{e}_t}{dt} = \dot{\theta}\left(1+\theta^2\right)^{-3/2}(-\theta\mathbf{e}_r + \mathbf{e}_\theta) \tag{2.256}$$

Next, the second term in Eq. (2.254) is obtained as

$${}^{\mathcal{F}}\boldsymbol{\omega}^{\mathcal{A}} \times \mathbf{e}_t = \dot{\theta}\mathbf{E}_z \times \left(1+\theta^2\right)^{-1/2}(\mathbf{e}_r + \theta\mathbf{e}_\theta) \tag{2.257}$$

Eq. (2.257) simplifies to

$${}^{\mathcal{F}}\boldsymbol{\omega}^{\mathcal{A}} \times \mathbf{e}_t = \dot{\theta}(1+\theta^2)^{-1/2}(-\theta\mathbf{e}_r + \mathbf{e}_\theta) \tag{2.258}$$

Eq. (2.258) can be re-written as

$${}^{\mathcal{F}}\boldsymbol{\omega}^{\mathcal{A}} \times \mathbf{e}_t = \dot{\theta}(1+\theta^2)(1+\theta^2)^{-3/2}(-\theta\mathbf{e}_r + \mathbf{e}_\theta) \tag{2.259}$$

Then, adding Eq. (2.256) and Eq. (2.259), we obtain

$$\frac{{}^{\mathcal{F}}d\mathbf{e}_t}{dt} = \dot{\theta}\left(1+\theta^2\right)^{-3/2}\left(2+\theta^2\right)(-\theta\mathbf{e}_r + \mathbf{e}_\theta) \tag{2.260}$$

Then the magnitude of ${}^{\mathcal{F}}d\mathbf{e}_t/dt$ is obtained as

$$\left\|\frac{{}^{\mathcal{F}}d\mathbf{e}_t}{dt}\right\| = \dot{\theta}\left(1+\theta^2\right)^{-3/2}\left(2+\theta^2\right)\sqrt{1+\theta^2} \tag{2.261}$$

Then, dividing Eq. (2.260) by Eq. (2.261), we obtain the principle unit normal in reference frame $\mathcal{F}$ as

$$\mathbf{e}_n = \frac{-\theta\mathbf{e}_r + \mathbf{e}_\theta}{\sqrt{1+\theta^2}} \tag{2.262}$$

Finally, the principle unit bi-normal vector in reference frame $\mathcal{F}$ is obtained as

$$\mathbf{e}_b = \mathbf{e}_t \times \mathbf{e}_n = \frac{\mathbf{e}_r + \theta\mathbf{e}_\theta}{\sqrt{1+\theta^2}} \times \frac{-\theta\mathbf{e}_r + \mathbf{e}_\theta}{\sqrt{1+\theta^2}} = \mathbf{e}_z \tag{2.263}$$

Curvature of Trajectory in Reference Frame $\mathcal{F}$

First, we know that

$$\frac{{}^{\mathcal{F}}d\mathbf{e}_t}{dt} = \kappa {}^{\mathcal{F}}v\mathbf{e}_n \tag{2.264}$$

Taking the magnitude of both sides, we have that

$$\left\|\frac{{}^{\mathcal{F}}d\mathbf{e}_t}{dt}\right\| = \kappa {}^{\mathcal{F}}v \tag{2.265}$$

Solving for κ, we have that

$$\kappa = \frac{\|{}^{\mathcal{F}}d\mathbf{e}_t/dt\|}{{}^{\mathcal{F}}v} \tag{2.266}$$

Substituting the expression for $\|{}^{\mathcal{F}}d\mathbf{e}_t/dt\|$ from Eq. (2.261) and the expression for ${}^{\mathcal{F}}v$ from Eq. (2.250) into Eq. (2.266), we obtain κ as

$$\kappa = \frac{\dot\theta\left(1+\theta^2\right)^{-3/2}\left(2+\theta^2\right)\sqrt{1+\theta^2}}{a\dot\theta\sqrt{1+\theta^2}} = \frac{2+\theta^2}{a(1+\theta^2)^{3/2}} \tag{2.267}$$

Velocity and Acceleration of Particle

The velocity of the particle in reference frame $\mathcal{F}$ is given in intrinsic coordinates as

$${}^{\mathcal{F}}\mathbf{v} = {}^{\mathcal{F}}v\mathbf{e}_t \tag{2.268}$$

Using the expression for ${}^{\mathcal{F}}v$ from Eq. (2.250), we obtain ${}^{\mathcal{F}}\mathbf{v}$ as

$${}^{\mathcal{F}}\mathbf{v} = a\dot\theta\sqrt{1+\theta^2}\mathbf{e}_t \tag{2.269}$$

Furthermore, the acceleration in reference frame $\mathcal{F}$ is obtained in intrinsic coordinates as

$${}^{\mathcal{F}}\mathbf{a} = \frac{d}{dt}\left({}^{\mathcal{F}}v\right)\mathbf{e}_t + \kappa\left({}^{\mathcal{F}}v\right)^2\mathbf{e}_n \tag{2.270}$$

Differentiating ${}^{\mathcal{F}}v$ from Eq. (2.250), we have that

$$\frac{d}{dt}\left({}^{\mathcal{F}}v\right) = a\ddot\theta\sqrt{1+\theta^2} + a\dot\theta(1+\theta^2)^{-1/2}2\theta\dot\theta \tag{2.271}$$

Simplifying Eq. (2.271), we obtain

$$\frac{d}{dt}\left({}^{\mathcal{F}}v\right) = a\left(1+\theta^2\right)^{-1/2}\left[\ddot\theta\left(1+\theta^2\right)+\dot\theta^2\theta\right] \tag{2.272}$$

Next, using the expression for κ from Eq. (2.267), we have that

$$\kappa\left({}^{\mathcal{F}}v\right)^2 = \frac{2+\theta^2}{a(1+\theta^2)^{3/2}}\left(a\dot\theta\sqrt{1+\theta^2}\right) = \frac{a(2+\theta^2)\dot\theta^2}{\sqrt{1+\theta^2}} \tag{2.273}$$

Substituting the results of Eq. (2.272) and Eq. (2.273) into Eq. (2.270), we obtain the acceleration of the particle in reference frame $\mathcal{F}$ as

$$^{\mathcal{F}}\mathbf{a} = a\left(1+\theta^2\right)^{-1/2}\left[\ddot{\theta}\left(1+\theta^2\right)+\dot{\theta}^2\theta\right]\mathbf{e}_t + \frac{a(2+\theta^2)\dot{\theta}^2}{\sqrt{1+\theta^2}}\mathbf{e}_n \tag{2.274}$$

Simplifying Eq. (2.274) gives

$$^{\mathcal{F}}\mathbf{a} = \frac{a}{\sqrt{1+\theta^2}}\left[\left(\ddot{\theta}(1+\theta^2)+\dot{\theta}^2\theta\right)\mathbf{e}_t + (2+\theta^2)\dot{\theta}^2\mathbf{e}_n\right] \tag{2.275}$$

Question 2–19

A particle P slides without friction along the inside of a fixed hemispherical bowl of radius R as shown in Fig. P2-19. The basis $\{\mathbf{E}_x, \mathbf{E}_y, \mathbf{E}_z\}$ is fixed to the bowl. Furthermore, the angle θ is measured from the $\mathbf{E}_x$-direction to the direction OQ, where point Q lies on the rim of the bowl while the angle ϕ is measured from the OQ-direction to the position of the particle. Determine the velocity and acceleration of the particle as viewed by an observer fixed to the bowl. **Hint:** Express the position in terms of a spherical basis that is fixed to the direction OP; then determine the velocity and acceleration as viewed by an observer fixed to the bowl in terms of this spherical basis.

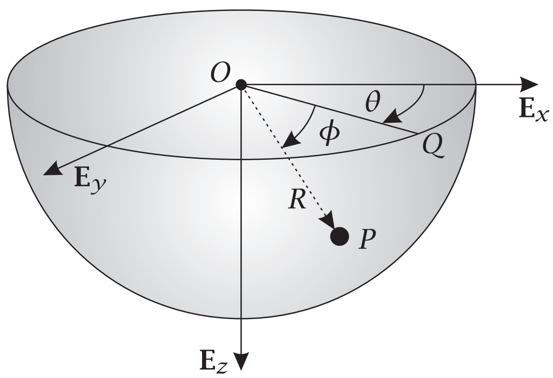

Figure P2-18

Solution to Question 2–19

Let $\mathcal{F}$ be a reference frame fixed to the bowl. Then choose the following coordinate system fixed in reference frame $\mathcal{F}$:

$$\begin{array}{ccl} & \text{Origin at } O & \\ \mathbf{E}_x & = & \text{Given} \\ \mathbf{E}_y & = & \text{Given} \\ \mathbf{E}_z & = & \mathbf{E}_x \times \mathbf{E}_y = \text{Given} \end{array}$$

Next, let $\mathcal{A}$ be a reference frame fixed to the plane defined by the points O and Q and the direction $\mathbf{E}_z$. Then choose the following coordinate system fixed in reference frame $\mathcal{A}$:

$$\begin{array}{ccl} & \text{Origin at } O & \\ \mathbf{e}_r & = & \text{Along } OQ \\ \mathbf{e}_z & = & \mathbf{E}_z \\ \mathbf{e}_\theta & = & \mathbf{e}_z \times \mathbf{e}_r \end{array}$$

Finally, let $\mathcal{B}$ be a reference frame fixed to the direction OP. Then choose the following coordinate system fixed in reference frame $\mathcal{B}$:

$$\begin{array}{ccc} & \text{Origin at } O & \\ \mathbf{u}_r & = & \text{Along } OP \\ \mathbf{u}_\theta & = & \mathbf{e}_\theta \\ \mathbf{u}_\phi & = & \mathbf{u}_r \times \mathbf{u}_\theta \end{array}$$

The relationship between the bases $\{\mathbf{E}_x, \mathbf{E}_y, \mathbf{E}_z\}$ and $\{\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_z\}$ is shown in Fig. 2-5 while the relationship between the bases $\{\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_z\}$ and $\{\mathbf{u}_r, \mathbf{u}_\theta, \mathbf{u}_\phi\}$ is shown in Fig. 2-6.

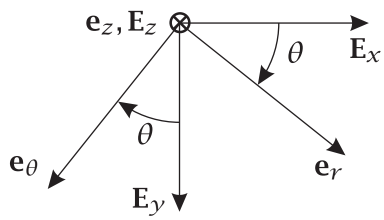

Figure 2-5 Relationship between bases $\{\mathbf{E}_x, \mathbf{E}_y, \mathbf{E}_z\}$ and $\{\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_z\}$ for Question 2–19.

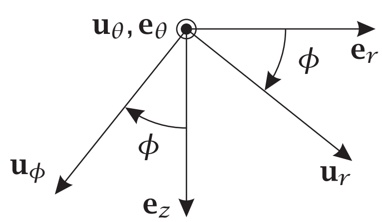

Figure 2-6 Relationship between bases $\{\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_z\}$ and $\{\mathbf{u}_r, \mathbf{u}_\theta, \mathbf{u}_\phi\}$ for Question 2–19.

The position of the particle is then given as

$$\mathbf{r} = R\mathbf{u}_r \tag{2.276}$$

Next, the angular velocity of reference frame $\mathcal{A}$ in reference frame $\mathcal{F}$ is given as

$${}^{\mathcal{F}}\boldsymbol{\omega}^{\mathcal{A}} = \dot{\theta}\mathbf{e}_z \tag{2.277}$$

Furthermore, the angular velocity of reference frame $\mathcal{B}$ in reference frame $\mathcal{A}$ is given as

$${}^{\mathcal{A}}\boldsymbol{\omega}^{\mathcal{B}} = -\dot{\phi}\mathbf{u}_\theta \tag{2.278}$$

where the negative sign on ${}^{\mathcal{A}}\boldsymbol{\omega}^{\mathcal{B}}$ is due to the fact that the angle ϕ is measured positively about the *negative* $\mathbf{u}_\theta$-direction (see Fig. 2-6). Then, applying the angular velocity addition theorem, we have

$${}^{\mathcal{F}}\boldsymbol{\omega}^{\mathcal{B}} = {}^{\mathcal{F}}\boldsymbol{\omega}^{\mathcal{A}} + {}^{\mathcal{A}}\boldsymbol{\omega}^{\mathcal{B}} = \dot{\theta}\mathbf{e}_z - \dot{\phi}\mathbf{u}_\theta \tag{2.279}$$

Now we can obtain an expression for ${}^{\mathcal{F}}\boldsymbol{\omega}^{\mathcal{B}}$ in terms of the basis $\{\mathbf{u}_r, \mathbf{u}_\theta, \mathbf{u}_\phi\}$ by expressing $\mathbf{e}_z$ in terms of $\mathbf{u}_r$ and $\mathbf{u}_\phi$ as

$$\mathbf{e}_z = \sin\phi\mathbf{u}_r + \cos\phi\mathbf{u}_\phi \tag{2.280}$$

Consequently,

$${}^{\mathcal{F}}\boldsymbol{\omega}^{\mathcal{B}} = \dot{\theta}(\sin\phi\mathbf{u}_r + \cos\phi\mathbf{u}_\phi) - \dot{\phi}\mathbf{u}_\theta = \dot{\theta}\sin\phi\mathbf{u}_r - \dot{\phi}\mathbf{u}_\theta + \dot{\theta}\cos\phi\mathbf{u}_\phi \tag{2.281}$$

Then, the velocity of point P in reference frame $\mathcal{F}$ is obtained by applying the transport theorem between reference frames $\mathcal{B}$ and $\mathcal{F}$ as

$${}^{\mathcal{F}}\mathbf{v} = \frac{{}^{\mathcal{F}}d\mathbf{r}}{dt} = \frac{{}^{\mathcal{B}}d\mathbf{r}}{dt} + {}^{\mathcal{F}}\boldsymbol{\omega}^{\mathcal{B}} \times \mathbf{r} \tag{2.282}$$

Now we have

$$\begin{aligned} \frac{{}^{\mathcal{B}}d\mathbf{r}}{dt} &= \mathbf{0} && (2.283)\\ {}^{\mathcal{F}}\boldsymbol{\omega}^{\mathcal{B}} \times \mathbf{r} &= (\dot{\theta}\sin\phi\mathbf{u}_r - \dot{\phi}\mathbf{u}_\theta + \dot{\theta}\cos\phi\mathbf{u}_\phi) \times R\mathbf{u}_r \\ &= R\dot{\theta}\cos\phi\mathbf{u}_\theta + R\dot{\phi}\mathbf{u}_\phi && (2.284) \end{aligned}$$

Therefore,

$${}^{\mathcal{F}}\mathbf{v} = R\dot{\theta}\cos\phi\mathbf{u}_\theta + R\dot{\phi}\mathbf{u}_\phi \tag{2.285}$$

The acceleration of point P in reference frame $\mathcal{F}$ is obtained by applying the transport theorem to ${}^{\mathcal{F}}\mathbf{v}$ between reference frames $\mathcal{B}$ and $\mathcal{F}$ as

$${}^{\mathcal{F}}\mathbf{a} = \frac{{}^{\mathcal{F}}d}{dt}\left({}^{\mathcal{F}}\mathbf{v}\right) = \frac{{}^{\mathcal{B}}d}{dt}\left({}^{\mathcal{F}}\mathbf{v}\right) + {}^{\mathcal{F}}\boldsymbol{\omega}^{\mathcal{B}} \times {}^{\mathcal{F}}\mathbf{v} \tag{2.286}$$

Now we have

$$\begin{aligned} \frac{{}^{\mathcal{B}}d}{dt}\left({}^{\mathcal{F}}\mathbf{v}\right) &= R(\ddot{\theta}\cos\phi - \dot{\theta}\dot{\phi}\sin\phi)\mathbf{u}_\theta + R\ddot{\phi}\mathbf{u}_\phi && (2.287)\\ {}^{\mathcal{F}}\boldsymbol{\omega}^{\mathcal{B}} \times {}^{\mathcal{F}}\mathbf{v} &= (\dot{\theta}\sin\phi\mathbf{u}_r - \dot{\phi}\mathbf{u}_\theta + \dot{\theta}\cos\phi\mathbf{u}_\phi) \times (R\dot{\theta}\cos\phi\mathbf{u}_\theta + R\dot{\phi}\mathbf{u}_\phi) \\ &= R\dot{\theta}^2\cos\phi\sin\phi\mathbf{u}_\phi - R\dot{\theta}\dot{\phi}\sin\phi\mathbf{u}_\theta \\ &\quad -R\dot{\phi}^2\mathbf{u}_r - R\dot{\theta}^2\cos^2\phi\mathbf{u}_r && (2.288) \end{aligned}$$

Adding these last two equations and simplifying gives

$$\begin{aligned} {}^{\mathcal{F}}\mathbf{a} = &-(R\dot{\phi}^2 + R\dot{\theta}^2\cos^2\phi)\mathbf{u}_r \\ &+ (R\ddot{\theta}\cos\phi - 2R\dot{\theta}\dot{\phi}\sin\phi)\mathbf{u}_\theta \\ &+ (R\ddot{\phi} + R\dot{\theta}^2\cos\phi\sin\phi)\mathbf{u}_\phi \end{aligned} \tag{2.289}$$

Question 2–20

A particle P slides along a circular table as shown in Fig. P2-20. The table is rigidly attached to two shafts such that the shafts and table rotate with angular velocity $\boldsymbol{\Omega}$ about an axis along the direction of the shafts. Knowing that the position of the particle is given in terms of a polar coordinate system relative to the table, determine (a) the angular velocity of the table as viewed by an observer fixed to the ground, (b) the velocity and acceleration of the particle as viewed by an observer fixed to the table, and (c) the velocity and acceleration of the particle as viewed by an observer fixed to the ground.

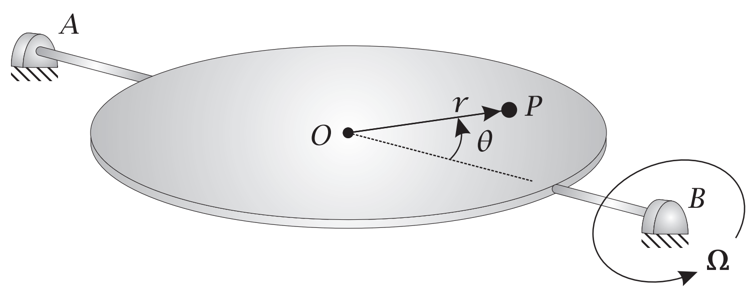

Figure P2-19

Solution to Question 2–20

Let $\mathcal{F}$ be a reference frame fixed to the ground. Then choose the following coordinate system fixed in reference frame $\mathcal{F}$:

$$
\begin{array}{lcl}
 & \text{Origin at 0} & \\
\mathbf{E}_x & = & \text{Along } OB \\
\mathbf{E}_z & = & \text{Vertically Upward} \\
\mathbf{E}_y & = & \mathbf{E}_z \times \mathbf{E}_x
\end{array}
$$

Next, let $\mathcal{A}$ be a reference frame fixed to the table. Then choose the following coordinate system fixed in reference frame $\mathcal{A}$:

$$
\begin{array}{lcl}
 & \text{Origin at 0} & \\
\mathbf{e}_x & = & \text{Along } OB \\
\mathbf{e}_z & = & \text{Orthogonal to Table and} = \mathbf{E}_z \text{ When } t = 0 \\
\mathbf{e}_y & = & \mathbf{e}_z \times \mathbf{e}_x
\end{array}
$$

Finally, let $\mathcal{B}$ be a reference frame fixed to the direction of OP. Then choose the following coordinate system fixed in reference frame $\mathcal{B}$:

$$
\begin{array}{lcl}
 & \text{Origin at 0} & \\
\mathbf{e}_r & = & \text{Along } OB \\
\mathbf{e}_z & = & \text{Same as in Reference Frame } \mathcal{B} \\
\mathbf{e}_\theta & = & \mathbf{e}_z \times \mathbf{e}_r
\end{array}
$$

The position of the particle is then given as

$$\mathbf{r} = r\mathbf{e}_r \tag{2.290}$$

Now because the position is expressed in terms of the basis $\{\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_z\}$ and $\{\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_z\}$ is fixed in reference frame $\mathcal{B}$, the velocity of the particle as viewed by an observer fixed to the ground is obtained by applying the transport theorem between reference frames $\mathcal{B}$ and $\mathcal{F}$ as

$$^{\mathcal{F}}\mathbf{v} = \frac{^{\mathcal{F}}d\mathbf{r}}{dt} = \frac{^{\mathcal{B}}d\mathbf{r}}{dt} + {^{\mathcal{F}}}\boldsymbol{\omega}^{\mathcal{B}} \times \mathbf{r} \tag{2.291}$$

First, the angular velocity of $\mathcal{B}$ in $\mathcal{F}$ is obtained from the angular velocity addition theorem as

$$^{\mathcal{F}}\boldsymbol{\omega}^{\mathcal{B}} = {^{\mathcal{F}}}\boldsymbol{\omega}^{\mathcal{A}} + {^{\mathcal{A}}}\boldsymbol{\omega}^{\mathcal{B}} \tag{2.292}$$

Now we have

$$\begin{aligned} ^{\mathcal{F}}\boldsymbol{\omega}^{\mathcal{A}} &= \boldsymbol{\Omega} = \Omega\mathbf{e}_x && (2.293) \\ ^{\mathcal{A}}\boldsymbol{\omega}^{\mathcal{B}} &= \dot{\theta}\mathbf{e}_z && (2.294) \end{aligned}$$

which implies that

$$^{\mathcal{F}}\boldsymbol{\omega}^{\mathcal{B}} = \Omega\mathbf{e}_x + \dot{\theta}\mathbf{e}_z \tag{2.295}$$

Next, because the position is expressed in terms of the basis $\{\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_z\}$, the unit vector $\mathbf{e}_x$ must also be expressed in terms of the basis $\{\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_z\}$. The relationship between the bases $\{\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z\}$ and $\{\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_z\}$ is shown in Fig. 2-7. Using Fig. 2-7, it is seen that

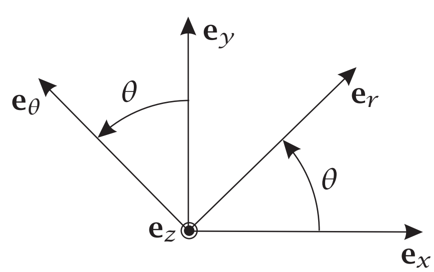

Figure 2-7 Geometry of bases $\{\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z\}$ and $\{\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_z\}$ for Question P2–20.

$$\begin{aligned} \mathbf{e}_x &= \cos\theta\,\mathbf{e}_r - \sin\theta\,\mathbf{e}_\theta && (2.296) \\ \mathbf{e}_y &= \sin\theta\,\mathbf{e}_r + \cos\theta\,\mathbf{e}_\theta && (2.297) \end{aligned}$$

Therefore,

$$^{\mathcal{F}}\boldsymbol{\omega}^{\mathcal{B}} = \Omega(\cos\theta\,\mathbf{e}_r - \sin\theta\,\mathbf{e}_\theta) + \dot{\theta}\mathbf{e}_z = \Omega\cos\theta\,\mathbf{e}_r - \Omega\sin\theta\,\mathbf{e}_\theta + \dot{\theta}\mathbf{e}_z \tag{2.298}$$

Now the two terms required to obtain ${}^{\mathcal{F}}\mathbf{v}$ are given as

$$\frac{{}^{\mathcal{F}}d\mathbf{r}}{dt} = \dot{r}\mathbf{e}_r \tag{2.299}$$

$${}^{\mathcal{F}}\boldsymbol{\omega}^{\mathcal{B}} \times \mathbf{r} = (\Omega\cos\theta\,\mathbf{e}_r - \Omega\sin\theta\,\mathbf{e}_\theta + \dot{\theta}\mathbf{e}_z) \times r\mathbf{e}_r = r\dot{\theta}\mathbf{e}_\theta + r\Omega\sin\theta\,\mathbf{e}_z \tag{2.300}$$

Therefore, the velocity of the particle in reference frame $\mathcal{F}$ is

$${}^{\mathcal{F}}\mathbf{v} = \dot{r}\mathbf{e}_r + r\dot{\theta}\mathbf{e}_\theta + r\Omega\sin\theta\,\mathbf{e}_z \tag{2.301}$$

Next, the acceleration of the particle as viewed by an observer fixed to the ground is given from the transport theorem as

$${}^{\mathcal{F}}\mathbf{a} = \frac{{}^{\mathcal{F}}d}{dt}\left({}^{\mathcal{F}}\mathbf{v}\right) = \frac{{}^{\mathcal{B}}d}{dt}\left({}^{\mathcal{F}}\mathbf{v}\right) + {}^{\mathcal{F}}\boldsymbol{\omega}^{\mathcal{B}} \times {}^{\mathcal{F}}\mathbf{v} \tag{2.302}$$

Now we have

$$\frac{{}^{\mathcal{B}}d}{dt}\left({}^{\mathcal{F}}\mathbf{v}\right) = \ddot{r}\mathbf{e}_r + (\dot{r}\dot{\theta} + r\ddot{\theta})\mathbf{e}_\theta + \left[\dot{r}\Omega\sin\theta + r(\dot{\Omega}\sin\theta + \Omega\dot{\theta}\cos\theta)\right]\mathbf{e}_z \tag{2.303}$$

$$\begin{aligned}
{}^{\mathcal{F}}\boldsymbol{\omega}^{\mathcal{B}} \times {}^{\mathcal{F}}\mathbf{v} &= (\Omega\cos\theta\,\mathbf{e}_r - \Omega\sin\theta\,\mathbf{e}_\theta + \dot{\theta}\mathbf{e}_z) \times (\dot{r}\mathbf{e}_r + r\dot{\theta}\mathbf{e}_\theta + r\Omega\sin\theta\,\mathbf{e}_z) \\
&= r\dot{\theta}\Omega\cos\theta\,\mathbf{e}_z - r\Omega^2\cos\theta\,\sin\theta\,\mathbf{e}_\theta + \dot{r}\Omega\sin\theta\,\mathbf{e}_z \\
&\quad -r\Omega^2\sin^2\theta\mathbf{e}_r + \dot{r}\dot{\theta}\mathbf{e}_\theta - r\dot{\theta}^2\mathbf{e}_r \\
&= -(r\dot{\theta}^2 + r\Omega^2\sin^2\theta)\mathbf{e}_r + (\dot{r}\dot{\theta} - r\Omega^2\cos\theta\,\sin\theta)\mathbf{e}_\theta \\
&\quad + (\dot{r}\Omega\sin\theta + r\dot{\theta}\Omega\cos\theta)\mathbf{e}_z
\end{aligned} \tag{2.304}$$

Adding these last two equations, we obtain the acceleration as viewed by an observer fixed to the ground as

$$\begin{aligned}
{}^{\mathcal{F}}\mathbf{a} &= (\ddot{r} - r\dot{\theta}^2 - r\Omega^2\sin^2\theta)\mathbf{e}_r \\
&\quad + (2\dot{r}\dot{\theta} + r\ddot{\theta} - r\Omega^2\cos\theta\,\sin\theta)\mathbf{e}_\theta \\
&\quad + \left[r\dot{\Omega}\sin\theta + 2(\dot{r}\Omega\sin\theta + r\Omega\dot{\theta}\cos\theta)\right]\mathbf{e}_z
\end{aligned} \tag{2.305}$$

Question 2–21

A slender rod of length l is hinged to a collar as shown in Fig. P2-21. The collar slides freely along a fixed horizontal track. Knowing that x is the horizontal displacement of the collar and that θ describes the orientation of the rod relative to the vertical direction, determine the velocity and acceleration of the free end of the rod as viewed by an observer fixed to the track.

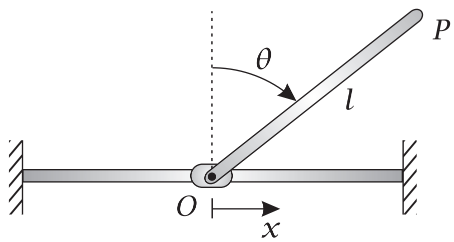

Figure P2-20

Solution to Question 2–21

Let $\mathcal{F}$ be a reference frame fixed to the horizontal track. Then, choose the following coordinate system fixed in reference frame $\mathcal{F}$:

	Origin at O at $t = 0$	
$\mathbf{E}_x$	$=$	To the Right
$\mathbf{E}_z$	$=$	Into Page
$\mathbf{E}_y$	$=$	$\mathbf{E}_z \times \mathbf{E}_x$

Next, let $\mathcal{A}$ be a reference frame fixed to the rod. Then, choose the following coordinate system fixed in reference frame $\mathcal{A}$:

	Origin at O	
$\mathbf{e}_r$	$=$	Along OP
$\mathbf{e}_z$	$=$	Into Page
$\mathbf{e}_\theta$	$=$	$\mathbf{E}_z \times \mathbf{e}_r$

We note that the relationship between the basis $\{\mathbf{E}_x, \mathbf{E}_y, \mathbf{E}_z\}$ and $\{\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_z\}$ is given as

$$\begin{aligned} \mathbf{E}_x &= \sin\theta\,\mathbf{e}_r + \cos\theta\,\mathbf{e}_\theta \\ \mathbf{E}_y &= -\cos\theta\,\mathbf{e}_r + \sin\theta\,\mathbf{e}_\theta \end{aligned} \tag{2.306}$$

Using the bases $\{\mathbf{E}_x, \mathbf{E}_y, \mathbf{E}_z\}$ and $\{\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_z\}$, the position of point P is given as

$$\mathbf{r}_P = \mathbf{r}_O + \mathbf{r}_{P/O} = x\mathbf{E}_x + l\mathbf{e}_r \tag{2.307}$$

Next, the angular velocity of reference frame $\mathcal{A}$ in reference frame $\mathcal{F}$ is given as

$$ {}^{\mathcal{F}}\boldsymbol{\omega}^{\mathcal{A}} = \dot{\theta}\mathbf{e}_z \tag{2.308} $$

The velocity of point P in reference frame $\mathcal{F}$ is then given as

$$ {}^{\mathcal{F}}\mathbf{v}_P = \frac{{}^{\mathcal{F}}d}{dt}\left(\mathbf{r}_O\right) + \frac{{}^{\mathcal{F}}d}{dt}\left(\mathbf{r}_{P/O}\right) = {}^{\mathcal{F}}\mathbf{v}_O + {}^{\mathcal{F}}\mathbf{v}_{P/O} \tag{2.309} $$

Now since $\mathbf{r}_O$ is expressed in the basis $\{\mathbf{E}_x, \mathbf{E}_y, \mathbf{E}_z\}$ and $\{\mathbf{E}_x, \mathbf{E}_y, \mathbf{E}_z\}$ is fixed, we have that

$$ {}^{\mathcal{F}}\mathbf{v}_O = \frac{{}^{\mathcal{F}}d}{dt}\left(\mathbf{r}_O\right) = \dot{x}\mathbf{E}_x \tag{2.310} $$

Next, since $\mathbf{r}_{P/O}$ is expressed in the basis $\{\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_z\}$ and $\{\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_z\}$ rotates with angular velocity ${}^{\mathcal{F}}\boldsymbol{\omega}^{\mathcal{A}}$, we can apply the rate of change transport theorem to $\mathbf{r}_{P/O}$ between reference frame $\mathcal{A}$ and reference frame $\mathcal{F}$ as

$$ {}^{\mathcal{F}}\mathbf{v}_{P/O} = \frac{{}^{\mathcal{F}}d}{dt}\left(\mathbf{r}_{P/O}\right) = \frac{{}^{\mathcal{A}}d}{dt}\left(\mathbf{r}_{P/O}\right) + {}^{\mathcal{F}}\boldsymbol{\omega}^{\mathcal{A}} \times \mathbf{r}_{P/O} \tag{2.311} $$

Now we have that

$$ \frac{{}^{\mathcal{A}}d}{dt}\left(\mathbf{r}_{P/O}\right) = \mathbf{0} \tag{2.312} $$

$$ {}^{\mathcal{F}}\boldsymbol{\omega}^{\mathcal{A}} \times \mathbf{r}_{P/O} = \dot{\theta}\mathbf{e}_z \times l\mathbf{e}_r = l\dot{\theta}\mathbf{e}_\theta \tag{2.313} $$

Adding Eq. (2.312) and Eq. (2.313) gives

$$ {}^{\mathcal{F}}\mathbf{v}_{P/O} = l\dot{\theta}\mathbf{e}_\theta \tag{2.314} $$

Therefore, the velocity of point P in reference frame $\mathcal{F}$ is given as

$$ {}^{\mathcal{F}}\mathbf{v}_P = \dot{x}\mathbf{E}_x + l\dot{\theta}\mathbf{e}_\theta \tag{2.315} $$

Next, the acceleration of point P in reference frame $\mathcal{F}$ is obtained as

$$ {}^{\mathcal{F}}\mathbf{a}_P = \frac{{}^{\mathcal{F}}d}{dt}\left({}^{\mathcal{F}}\mathbf{v}_P\right) \tag{2.316} $$

Now we have that

$$ {}^{\mathcal{F}}\mathbf{v}_P = {}^{\mathcal{F}}\mathbf{v}_O + {}^{\mathcal{F}}\mathbf{v}_{P/O} \tag{2.317} $$

where

$$ {}^{\mathcal{F}}\mathbf{v}_O = \dot{x}\mathbf{E}_x \tag{2.318} $$

$$ {}^{\mathcal{F}}\mathbf{v}_{P/O} = l\dot{\theta}\mathbf{e}_\theta \tag{2.319} $$

Consequently,

$$ {}^{\mathcal{F}}\mathbf{a}_P = \frac{{}^{\mathcal{F}}d}{dt}\left({}^{\mathcal{F}}\mathbf{v}_O\right) + \frac{{}^{\mathcal{F}}d}{dt}\left({}^{\mathcal{F}}\mathbf{v}_{P/O}\right) \tag{2.320} $$

Now, since ${}^{\mathcal{F}}\mathbf{v}_O$ is expressed in the basis $\{\mathbf{E}_x, \mathbf{E}_y, \mathbf{E}_z\}$, we have that

$$ {}^{\mathcal{F}}\mathbf{a}_O = \frac{{}^{\mathcal{F}}d}{dt}\left({}^{\mathcal{F}}\mathbf{v}_O\right) = \ddot{x}\mathbf{E}_x \tag{2.321} $$

Furthermore, since ${}^{\mathcal{F}}\mathbf{v}_{P/O}$ is expressed in the basis $\{\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_z\}$ and $\{\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_z\}$ rotates with angular velocity ${}^{\mathcal{F}}\boldsymbol{\omega}^{\mathcal{A}}$, we can obtain ${}^{\mathcal{F}}\mathbf{a}_{P/O}$ by applying the rate of change transport theorem between reference frame $\mathcal{A}$ and reference frame $\mathcal{F}$ as

$$ {}^{\mathcal{F}}\mathbf{a}_{P/O} = \frac{{}^{\mathcal{F}}d}{dt}\left({}^{\mathcal{F}}\mathbf{v}_O\right) = \frac{{}^{\mathcal{A}}d}{dt}\left({}^{\mathcal{F}}\mathbf{v}_O\right) + {}^{\mathcal{F}}\boldsymbol{\omega}^{\mathcal{A}} \times {}^{\mathcal{F}}\mathbf{v}_O \tag{2.322} $$

Now we have that

$$ \frac{{}^{\mathcal{A}}d}{dt}\left({}^{\mathcal{F}}\mathbf{v}_O\right) = l\ddot{\theta}\mathbf{e}_\theta \tag{2.323} $$

$$ {}^{\mathcal{F}}\boldsymbol{\omega}^{\mathcal{A}} \times {}^{\mathcal{F}}\mathbf{v}_O = \dot{\theta}\mathbf{e}_z \times l\dot{\theta}\mathbf{e}_\theta = -l\dot{\theta}^2\mathbf{e}_r \tag{2.324} $$

Adding Eq. (2.323) and Eq. (2.324) gives

$$ {}^{\mathcal{F}}\mathbf{a}_{P/O} = -l\dot{\theta}^2\mathbf{e}_r + l\ddot{\theta}\mathbf{e}_\theta \tag{2.325} $$

Then, adding Eq. (2.321) and Eq. (2.325), we obtain the velocity of point P in reference frame $\mathcal{F}$ as

$$ {}^{\mathcal{F}}\mathbf{a}_P = \ddot{x}\mathbf{E}_x - l\dot{\theta}^2\mathbf{e}_r + l\ddot{\theta}\mathbf{e}_\theta \tag{2.326} $$

Finally, substituting the expression for $\mathbf{E}_x$ from Eq. (2.306), we obtain ${}^{\mathcal{F}}\mathbf{a}_P$ in terms of the basis $\{\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_z\}$ as

$$ {}^{\mathcal{F}}\mathbf{a}_P = \ddot{x}(\sin\theta\,\mathbf{e}_r + \cos\theta\,\mathbf{e}_\theta) - l\dot{\theta}^2\mathbf{e}_r + l\ddot{\theta}\mathbf{e}_\theta = (\ddot{x}\sin\theta - l\dot{\theta}^2)\mathbf{e}_r + (\ddot{x}\cos\theta + l\ddot{\theta})\mathbf{e}_\theta \tag{2.327} $$

Question 2–23

A particle slides along a fixed track $y = -\ln\cos x$ as shown in Fig. P2-23 (where $-\pi/2 < x < \pi/2$). Using the horizontal component of position, x, as the variable to describe the motion and the initial condition $x(t = 0) = x_0$, determine the following quantities as viewed by an observer fixed to the track: (a) the arc-length parameter s as a function of x, (b) the intrinsic basis $\{\mathbf{e}_t, \mathbf{e}_n, \mathbf{e}_b\}$ and the curvature κ, and (c) the velocity and acceleration of the particle.

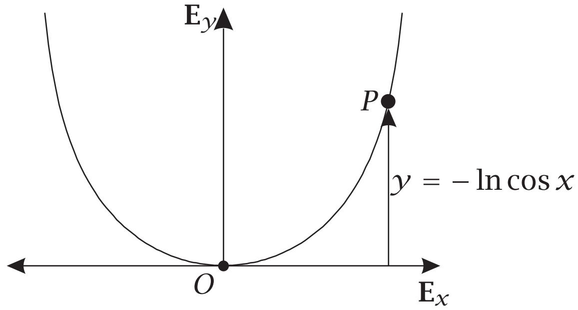

Figure P2-21

Solution to Question 2–23

For this problem, it is convenient to use a reference frame $\mathcal{F}$ that is fixed to the track. Then, we choose the following coordinate system fixed in reference frame $\mathcal{F}$:

$$\begin{array}{ccc} & \text{Origin at } O & \\ \mathbf{E}_x & = & \text{Along } Ox \\ \mathbf{E}_y & = & \text{Along } Oy \\ \mathbf{E}_z & = & \mathbf{E}_x \times \mathbf{E}_y \end{array}$$

The position of the particle is then given as

$$\mathbf{r} = x\mathbf{E}_x - \ln\cos x\mathbf{E}_y \tag{2.328}$$

Now, since the basis $\{\mathbf{E}_x, \mathbf{E}_y, \mathbf{E}_z\}$ does not rotate, the velocity in reference frame $\mathcal{F}$ is given as

$${}^{\mathcal{F}}\mathbf{v} = \dot{x}\mathbf{E}_x + \dot{x}\tan x\mathbf{E}_y \tag{2.329}$$

Using the velocity from Eq. (2.329), the speed of the particle in reference frame $\mathcal{F}$ is given as

$${}^{\mathcal{F}}v = \|{}^{\mathcal{F}}\mathbf{v}\| = \dot{x}\sqrt{1 + \tan^2 x} = \dot{x}\sec x \tag{2.330}$$

Arc-length Parameter as a Function of x

Now we recall the arc-length equation as

$$\frac{d}{dt}\left({}^{\mathcal{F}}s\right) = {}^{\mathcal{F}}v = \dot{x}\sqrt{1+\tan^2 x} = \dot{x}\sec x \tag{2.331}$$

Separating variables in Eq. (2.331), we obtain

$${}^{\mathcal{F}}ds = \sec x\, dx \tag{2.332}$$

Integrating both sides of Eq. (2.332) gives

$${}^{\mathcal{F}}s - {}^{\mathcal{F}}s_0 = \int_{x_0}^{x} \sec x\, dx \tag{2.333}$$

Using the integral given for $\sec x$, we obtain

$${}^{\mathcal{F}}s - {}^{\mathcal{F}}s_0 = \ln\left[\sec x + \tan x\right]_{x_0}^{x} = \ln\left[\frac{\sec x + \tan x}{\sec x_0 + \tan x_0}\right] \tag{2.334}$$

Noting that ${}^{\mathcal{F}}s(0) = {}^{\mathcal{F}}s_0 = 0$, the arc-length is given as

$${}^{\mathcal{F}}s = \ln\left[\sec x + \tan x\right]_{x_0}^{x} \tag{2.335}$$

Simplifying Eq. (2.335), we obtain

$${}^{\mathcal{F}}s = \ln\left[\frac{\sec x + \tan x}{\sec x_0 + \tan x_0}\right] \tag{2.336}$$

Intrinsic Basis

Next, we need to compute the intrinsic basis. First, we have the tangent vector as

$$\mathbf{e}_t = \frac{{}^{\mathcal{F}}\mathbf{v}}{{}^{\mathcal{F}}v} = \frac{\dot{x}(\mathbf{E}_x + \tan x\mathbf{E}_y)}{\dot{x}\sec x} = \frac{1}{\sec x}\mathbf{E}_x + \frac{\tan x}{\sec x}\mathbf{E}_y \tag{2.337}$$

Now we note that $\sec x = 1/\cos x$. Therefore,

$$\frac{\tan x}{\sec x} = \sin x \tag{2.338}$$

Eq. (2.337) then simplifies to

$$\mathbf{e}_t = \cos x\mathbf{E}_x + \sin x\mathbf{E}_y \tag{2.339}$$

Next, the principle unit normal is given as

$$\frac{{}^{\mathcal{F}}d\mathbf{e}_t}{dt} = \kappa\, {}^{\mathcal{F}}v\,\mathbf{e}_n \tag{2.340}$$

Differentiating $\mathbf{e}_t$ in Eq. (2.339), we obtain

$$\frac{{}^{\mathcal{F}}d\mathbf{e}_t}{dt} = -\dot{x}\sin x\mathbf{E}_x + \dot{x}\cos x\mathbf{E}_y \tag{2.341}$$

Consequently,

$$\left\|\frac{{}^{\mathcal{F}}d\mathbf{e}_t}{dt}\right\| = \dot{x} = \kappa\,{}^{\mathcal{F}}v \tag{2.342}$$

which implies that

$$\mathbf{e}_n = \frac{{}^{\mathcal{F}}d\mathbf{e}_t/dt}{\left\|{}^{\mathcal{F}}d\mathbf{e}_t/dt\right\|} = \frac{-\dot{x}\sin x\mathbf{E}_x + \dot{x}\cos x\mathbf{E}_y}{\dot{x}} = -\sin x\mathbf{E}_x + \cos x\mathbf{E}_y \tag{2.343}$$

Then, using ${}^{\mathcal{F}}v$ from Eq. (2.330), we obtain the curvature as

$$\kappa = \frac{\dot{x}}{\dot{x}\sec x} = \frac{1}{\sec x} = \cos x \tag{2.344}$$

Finally, the principle unit bi-normal is given as

$$\mathbf{e}_b = \mathbf{e}_t \times \mathbf{e}_n = (\cos x\mathbf{E}_x + \sin x\mathbf{E}_y) \times (-\sin x\mathbf{E}_x + \cos x\mathbf{E}_y) = \mathbf{E}_z \tag{2.345}$$

Velocity and Acceleration in Terms of Intrinsic Basis

Using the speed from Eq. (2.330), the velocity of the particle in reference frame $\mathcal{F}$ is given in terms of the instrinsic basis as

$${}^{\mathcal{F}}\mathbf{v} = \dot{x}\sec x\mathbf{e}_t \tag{2.346}$$

Next, the acceleration is given in terms of the intrinsic basis as

$${}^{\mathcal{F}}\mathbf{a} = \frac{d}{dt}\left({}^{\mathcal{F}}v\right)\mathbf{e}_t + \kappa\left({}^{\mathcal{F}}v\right)\mathbf{e}_n \tag{2.347}$$

Now, using ${}^{\mathcal{F}}v$ from Eq. (2.330), we obtain $d({}^{\mathcal{F}}v)/dt$ as

$$\frac{d}{dt}\left({}^{\mathcal{F}}v\right) = \ddot{x}\sec x + \dot{x}^2\sec x\tan x = \sec x\left[\ddot{x} + \dot{x}^2\tan x\right] \tag{2.348}$$

Also, using κ from Eq. (2.344) we obtain

$$\kappa\left({}^{\mathcal{F}}v\right) = \cos x(\dot{x}\sec x)^2 = \dot{x}^2\sec x \tag{2.349}$$

The acceleration of the particle in reference frame $\mathcal{F}$ is then given as

$${}^{\mathcal{F}}\mathbf{a} = \sec x\left[\ddot{x} + \dot{x}^2\tan x\right]\mathbf{e}_t + \dot{x}^2\sec x\mathbf{e}_n \tag{2.350}$$