Chapter 3

Kinetics of Particles

Question 3–1

A particle of mass m moves in the vertical plane along a track in the form of a circle as shown in Fig. P3-1. The equation for the track is

$$r = r_0 \cos\theta$$

Knowing that gravity acts downward and assuming the initial conditions $\theta(t = 0) = 0$ and $\dot{\theta}(t = 0) = \dot{\theta}_0$, determine (a) the differential equation of motion for the particle and (b) the force exerted by the track on the particle as a function of θ.

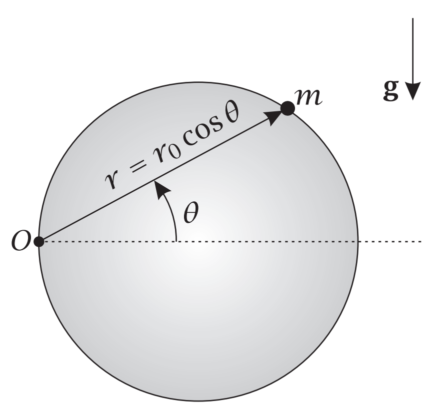


Figure P3-1

Solution to Question 3–1

Kinematics

Let $\mathcal{F}$ be a reference frame fixed to the track. Then, choose the following coordinate system fixed in reference frame $\mathcal{F}$:

$$
\begin{array}{lcl}
 & \text{Origin at point } O & \\
\mathbf{E}_x & = & \text{Along } OP \text{ when } \theta = 0 \\
\mathbf{E}_z & = & \text{Out of page} \\
\mathbf{E}_y & = & \mathbf{E}_z \times \mathbf{E}_x
\end{array}
$$

Next, let $\mathcal{A}$ be a reference frame fixed to the direction OP. Then, choose the following coordinate system fixed in reference frame $\mathcal{A}$:

$$
\begin{array}{lcl}
 & \text{Origin at point } O & \\
\mathbf{e}_r & = & \text{Along } OP \\
\mathbf{e}_z & = & \mathbf{E}_z \\
\mathbf{e}_\theta & = & \mathbf{e}_z \times \mathbf{e}_r
\end{array}
$$

The geometry of the bases $\{\mathbf{E}_x, \mathbf{E}_y, \mathbf{E}_z\}$ and $\{\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_z\}$ is shown in Fig. 3-1. Using Fig. 3-1, we have that

$$
\mathbf{E}_x = \cos\theta\,\mathbf{e}_r - \sin\theta\,\mathbf{e}_\theta \tag{3.1}
$$

$$
\mathbf{E}_y = \sin\theta\,\mathbf{e}_r + \cos\theta\,\mathbf{e}_\theta \tag{3.2}
$$

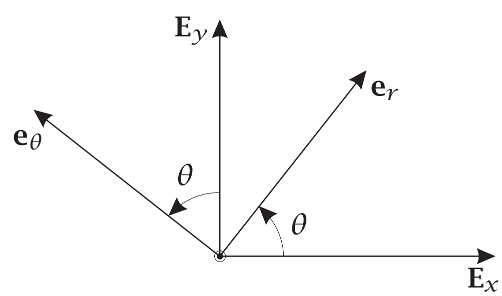


Figure 3-1 Geometry of Coordinate System for Question 3.1

Next, the position of the particle is given in terms of the basis $\{\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_z\}$ as

$$
\mathbf{r} = r\mathbf{e}_r = r_0\cos\theta\,\mathbf{e}_r \tag{3.3}
$$

Furthermore, since the angle θ is measured from the fixed horizontal direction, the angular velocity of $\mathcal{A}$ in $\mathcal{F}$ is given as

$$
{}^{\mathcal{F}}\boldsymbol{\omega}^{\mathcal{A}} = \dot{\theta}\mathbf{e}_z \tag{3.4}
$$

Applying the transport theorem to $\mathbf{r}$ from reference frame $\mathcal{A}$ to $\mathcal{F}$, the velocity of the particle in reference frame $\mathcal{F}$ as

$$^{\mathcal{F}}\mathbf{v} = \frac{^{\mathcal{F}}d\mathbf{r}}{dt} = \frac{^{\mathcal{A}}d\mathbf{r}}{dt} + {}^{\mathcal{F}}\boldsymbol{\omega}^{\mathcal{A}} \times \mathbf{r} \tag{3.5}$$

Now we have

$$\begin{aligned} \frac{^{\mathcal{A}}d\mathbf{r}}{dt} &= -r_0\dot{\theta}\sin\theta\,\mathbf{e}_r && (3.6) \\ {}^{\mathcal{F}}\boldsymbol{\omega}^{\mathcal{A}} \times \mathbf{r} &= \dot{\theta}\mathbf{E}_z \times r_0\cos\theta\,\mathbf{e}_r = r_0\dot{\theta}\cos\theta\,\mathbf{e}_\theta && (3.7) \end{aligned}$$

Adding the expressions in Eq. (3.6) and Eq. (3.7), we obtain the velocity in reference frame $\mathcal{F}$ as

$$^{\mathcal{F}}\mathbf{v} = -r_0\dot{\theta}\sin\theta\,\mathbf{e}_r + r_0\dot{\theta}\cos\theta\,\mathbf{e}_\theta \tag{3.8}$$

Re-writing Eq. (3.8), we obtain

$$^{\mathcal{F}}\mathbf{v} = r_0\dot{\theta}(-\sin\theta\,\mathbf{e}_r + \cos\theta\,\mathbf{e}_\theta) \tag{3.9}$$

The speed in reference frame $\mathcal{F}$ is then given as

$$^{\mathcal{F}}v = \|^{\mathcal{F}}\mathbf{v}\| = r_0\dot{\theta} \tag{3.10}$$

Dividing $^{\mathcal{F}}\mathbf{v}$ by $^{\mathcal{F}}v$, we obtain the tangent vector as

$$\mathbf{e}_t = -\sin\theta\,\mathbf{e}_r + \cos\theta\,\mathbf{e}_\theta \tag{3.11}$$

Next, the principal unit normal vector is computed as

$$\mathbf{e}_n = \frac{^{\mathcal{F}}d\mathbf{e}_t/dt}{\|^{\mathcal{F}}d\mathbf{e}_t/dt\|} \tag{3.12}$$

Applying the transport theorem to $\mathbf{e}_t$, we have

$$\frac{^{\mathcal{F}}d\mathbf{e}_t}{dt} = \frac{^{\mathcal{A}}d\mathbf{e}_t}{dt} + {}^{\mathcal{F}}\boldsymbol{\omega}^{\mathcal{A}} \times \mathbf{e}_t \tag{3.13}$$

Now

$$\begin{aligned} \frac{^{\mathcal{A}}d\mathbf{e}_t}{dt} &= -\dot{\theta}\cos\theta\,\mathbf{e}_r - \dot{\theta}\sin\theta\,\mathbf{e}_\theta && (3.14) \\ {}^{\mathcal{F}}\boldsymbol{\omega}^{\mathcal{A}} \times \mathbf{e}_t &= \dot{\theta}\mathbf{e}_z \times (-\sin\theta\,\mathbf{e}_r + \cos\theta\,\mathbf{e}_\theta) \\ &= -\dot{\theta}\cos\theta\,\mathbf{e}_r - \dot{\theta}\sin\theta\,\mathbf{e}_\theta && (3.15) \end{aligned}$$

Consequently,

$$\frac{^{\mathcal{F}}d\mathbf{e}_t}{dt} = -2\dot{\theta}\cos\theta\,\mathbf{e}_r - 2\dot{\theta}\sin\theta\,\mathbf{e}_\theta \tag{3.16}$$

which implies that

$$\mathbf{e}_n = \frac{-2\dot\theta\cos\theta\,\mathbf{e}_r - 2\dot\theta\sin\theta\,\mathbf{e}_\theta}{\|-2\dot\theta\cos\theta\,\mathbf{e}_r - 2\dot\theta\sin\theta\,\mathbf{e}_\theta\|} = -\cos\theta\,\mathbf{e}_r - \sin\theta\,\mathbf{e}_\theta \tag{3.17}$$

The principal unit bi-normal vector to the track is then obtained as

$$\mathbf{e}_b = \mathbf{e}_t \times \mathbf{e}_n = (-\sin\theta\,\mathbf{e}_r + \cos\theta\,\mathbf{e}_\theta) \times (-\cos\theta\,\mathbf{e}_r - \sin\theta\,\mathbf{e}_\theta) = \mathbf{e}_z \tag{3.18}$$

The acceleration as viewed by an observer fixed to the track is then obtained as

$${}^{\mathcal{F}}\mathbf{a} = \frac{{}^{\mathcal{F}}d}{dt}\left({}^{\mathcal{F}}\mathbf{v}\right) = \frac{{}^{\mathcal{A}}d}{dt}\left({}^{\mathcal{F}}\mathbf{v}\right) + {}^{\mathcal{F}}\boldsymbol{\omega}^{\mathcal{A}} \times {}^{\mathcal{F}}\mathbf{v} \tag{3.19}$$

Now we have

$$\begin{aligned}
\frac{{}^{\mathcal{A}}d}{dt}\left({}^{\mathcal{F}}\mathbf{v}\right) &= r_0\ddot\theta\mathbf{e}_t + r_0\dot\theta\,\frac{{}^{\mathcal{A}}d\mathbf{e}_t}{dt} \\
&= r_0\ddot\theta\mathbf{e}_t + r_0\dot\theta(-\dot\theta\cos\theta\,\mathbf{e}_r - \dot\theta\sin\theta\,\mathbf{e}_\theta) \\
&= r_0\ddot\theta\mathbf{e}_t + r_0\dot\theta^2(-\cos\theta\,\mathbf{e}_r - \sin\theta\,\mathbf{e}_\theta) \\
&= r_0\ddot\theta\mathbf{e}_t + r_0\dot\theta^2\mathbf{e}_n & (3.20) \\
{}^{\mathcal{F}}\boldsymbol{\omega}^{\mathcal{A}} \times {}^{\mathcal{F}}\mathbf{v} &= \dot\theta\mathbf{e}_z \times r_0\dot\theta\mathbf{e}_t \\
&= \dot\theta\mathbf{e}_b \times r_0\dot\theta\mathbf{e}_t = r_0\dot\theta^2\mathbf{e}_n & (3.21)
\end{aligned}$$

where we note that the results of Eqs. (3.14) and (3.17) have been used to obtain the result given in Eq. (3.20). Therefore,

$${}^{\mathcal{F}}\mathbf{a} = r_0\ddot\theta\mathbf{e}_t + 2r_0\dot\theta^2\mathbf{e}_n \tag{3.22}$$

Kinetics

Next, in order to obtain the differential equation of motion, we need to apply Newton's 2^{nd} Law to the particle. The free body diagram of the particle is given in Fig. 3-2 as where

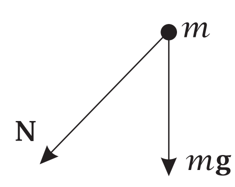


Figure 3-2 Free Body Diagram for Question 3–1.

$\mathbf{N}$	=	Reaction Force of Track on Particle
$m\mathbf{g}$	=	Force of Gravity

Now we know that the reaction force is orthogonal to the track while gravity acts vertically downward. Consequently, we have that

$$\begin{aligned} \mathbf{N} &= N_n \mathbf{e}_n + N_b \quad &(3.23) \\ m\mathbf{g} &= -mg\mathbf{E}_y \quad &(3.24) \end{aligned}$$

Then, using the expression for $\mathbf{E}_y$ from Eq. (3.2), we obtain the force of gravity as

$$m\mathbf{g} = -mg(\sin\theta\,\mathbf{e}_r + \cos\theta\,\mathbf{e}_\theta) = -mg\sin\theta\,\mathbf{e}_r - mg\cos\theta\,\mathbf{e}_\theta \quad (3.25)$$

The total force on the particle is then given as

$$\mathbf{F} = \mathbf{N} + m\mathbf{g} = N_n\mathbf{e}_n + N_b\mathbf{e}_b - mg\sin\theta\,\mathbf{e}_r - mg\cos\theta\,\mathbf{e}_\theta \quad (3.26)$$

Applying Newton's 2^{nd} Law using the acceleration from Eq. (3.22), we obtain

$$N_n\mathbf{e}_n + N_b\mathbf{e}_b - mg\sin\theta\,\mathbf{e}_r - mg\cos\theta\,\mathbf{e}_\theta = mr_0\ddot{\theta}\mathbf{e}_t + 2mr_0\dot{\theta}^2\mathbf{e}_n \quad (3.27)$$

Now it is seen that the unknown reaction forces exerted by the track lie in the directions of $\mathbf{e}_n$ and $\mathbf{e}_b$. Therefore, the reaction force exerted by the track can be eliminated if the scalar product with $\mathbf{e}_t$ is taken with both sides of Eq. (3.27) as

$$(N_n\mathbf{e}_n + N_b\mathbf{e}_b - mg\sin\theta\,\mathbf{e}_r - mg\cos\theta\,\mathbf{e}_\theta)\cdot\mathbf{e}_t = (mr_0\ddot{\theta}\mathbf{e}_t + 2mr_0\dot{\theta}^2\mathbf{e}_n)\cdot\mathbf{e}_t \quad (3.28)$$

Then, observing that $\mathbf{e}_n \cdot \mathbf{e}_t = \mathbf{e}_b \cdot \mathbf{e}_t = 0$, Eq. (3.28) simplifies to

$$-mg\sin\theta\,\mathbf{e}_r\cdot\mathbf{e}_t - mg\cos\theta\,\mathbf{e}_\theta\cdot\mathbf{e}_t = mr_0\ddot{\theta} \quad (3.29)$$

Now, using the expression for $\mathbf{e}_t$ from Eq. (3.11), we have that

$$\begin{aligned} \mathbf{e}_r\cdot\mathbf{e}_t &= \mathbf{e}_r\cdot(-\sin\theta\,\mathbf{e}_r + \cos\theta\,\mathbf{e}_\theta) = -\sin\theta \quad &(3.30) \\ \mathbf{e}_\theta\cdot\mathbf{e}_t &= \mathbf{e}_\theta\cdot(-\sin\theta\,\mathbf{e}_r + \cos\theta\,\mathbf{e}_\theta) = \cos\theta \quad &(3.31) \end{aligned}$$

Substituting the results of Eq. (3.30) and Eq. (3.31) into Eq. (3.29), we obtain

$$mg\sin^2\theta - mg\cos^2\theta = mr_0\ddot{\theta} \quad (3.32)$$

Now we also note that

$$\cos^2\theta - \sin^2\theta = \cos 2\theta \quad (3.33)$$

Therefore, Eq. (3.32) can be written as

$$-mg\cos 2\theta = mr_0\ddot{\theta} \quad (3.34)$$

Next, taking the scalar product of Eq. (3.28) in the $\mathbf{e}_n$ direction, we obtain

$$N_n - mg\sin\theta\,\mathbf{e}_r\cdot\mathbf{e}_n - mg\cos\theta\,\mathbf{e}_\theta\cdot\mathbf{e}_n = 2mr_0\dot{\theta}^2 \quad (3.35)$$

Then, using the expression for $\mathbf{e}_n$ from Eq. (3.17), we have that

$$\begin{aligned} \mathbf{e}_r \cdot \mathbf{e}_n &= \mathbf{e}_r \cdot (-\cos\theta\,\mathbf{e}_r - \sin\theta\,\mathbf{e}_\theta) = -\cos\theta \qquad (3.36) \\ \mathbf{e}_\theta \cdot \mathbf{e}_n &= \mathbf{e}_\theta \cdot (-\cos\theta\,\mathbf{e}_r - \sin\theta\,\mathbf{e}_\theta) = -\sin\theta \qquad (3.37) \end{aligned}$$

Substituting the results of Eq. (3.36) and Eq. (3.37) into Eq. (3.35) gives

$$N_n + mg\sin\theta\,\cos\theta + mg\cos\theta\,\sin\theta = 2mr_0\dot{\theta}^2 \qquad (3.38)$$

Now we have that

$$\sin\theta\,\cos\theta + \cos\theta\,\sin\theta = 2\sin\theta\,\cos\theta = \sin 2\theta \qquad (3.39)$$

Consequently, Eq. (3.38 can be written as

$$N + mg\sin 2\theta = 2mr_0\dot{\theta}^2 \qquad (3.40)$$

Finally, taking the scalar product of Eq. (3.28) in the $\mathbf{e}_b$ direction, we obtain

$$N_b = 0 \qquad (3.41)$$

The following three scalar equations then result from Eqs. (3.34), Eq. (3.40), and Eq. (3.41):

$$\begin{aligned} mr_0\ddot{\theta} &= -mg\cos 2\theta \qquad (3.42) \\ 2mr_0\dot{\theta}^2 &= N_n + mg\sin 2\theta \qquad (3.43) \\ 0 &= N_b \qquad (3.44) \end{aligned}$$

Since Eq. (3.43) contains no reaction forces, it is the differential equation of motion, i.e. the differential equation of motion is given as

$$mr_0\ddot{\theta} = -mg\cos 2\theta \qquad (3.45)$$

Rearranging this last equation, we obtain the differential equation of motion for the particle as

$$\ddot{\theta} + \frac{g}{r_0}\cos 2\theta = 0 \qquad (3.46)$$

Force Exerted by Track on Particle As a Function of θ

First we note the following:

$$\ddot{\theta} = \frac{d\dot{\theta}}{dt} = \frac{d\dot{\theta}}{d\theta}\frac{d\theta}{dt} = \dot{\theta}\frac{d\dot{\theta}}{d\theta} \qquad (3.47)$$

Substituting Eq. (3.47) into Eq. (3.46), we obtain

$$\dot{\theta}\frac{d\dot{\theta}}{d\theta} + \frac{g}{r_0}\cos 2\theta = 0 \qquad (3.48)$$

Rearranging Eq. (3.48) and separating variables, we obtain

$$\dot{\theta} d\dot{\theta} = -\frac{g}{r_0} \cos 2\theta d\theta \tag{3.49}$$

Integrating this last equation, we obtain

$$\frac{1}{2}\left(\dot{\theta}^2 - \dot{\theta}_0^2\right) = -\frac{g}{2r_0} \left[\sin 2\theta\right]_{\theta_0}^{\theta} \tag{3.50}$$

Noting that $\theta(t = 0) = 0$, this last equation simplifies to

$$\dot{\theta}^2 = \dot{\theta}_0^2 - \frac{g}{r_0} \sin 2\theta \tag{3.51}$$

Solving for the reaction force using Eq. (3.42), we obtain

$$N_n = 2mr_0\dot{\theta}^2 - mg \sin 2\theta \tag{3.52}$$

We then obtain

$$N_n = 2mr_0 \left[\dot{\theta}_0^2 - \frac{g}{r_0} \sin 2\theta\right] - mg \sin 2\theta \tag{3.53}$$

Simplifying this last equation, we obtain

$$N_n = 2mr_0\dot{\theta}_0^2 - 3mg \sin 2\theta \tag{3.54}$$

The force exerted by the track on the particle is then given as

$$\mathbf{N} = \left[2mr_0\dot{\theta}_0^2 - 3mg \sin 2\theta\right] \mathbf{e}_n \tag{3.55}$$

Question 3–2

A collar of mass m slides without friction along a rigid massless rod as shown in Fig. P3-2. The collar is attached to a linear spring with spring constant K and unstretched length L. Assuming no gravity, determine the differential equation of motion for the collar.

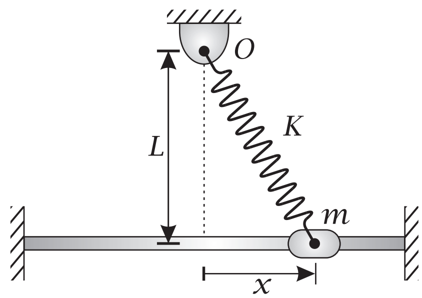


Figure P3-2

Solution to Question 3–2

First, let $\mathcal{F}$ be a fixed reference frame. Then, choose the following coordinate system fixed in reference frame $\mathcal{F}$:

$$\begin{array}{ccc} & \text{Origin at Attachment} & \\ & \text{Point of Spring} & \\ \mathbf{E}_y & = & \text{Up} \\ \mathbf{E}_z & = & \text{Out of Page} \\ \mathbf{E}_x & = & \mathbf{E}_y \times \mathbf{E}_z \end{array}$$

Then, in terms of the basis $\{\mathbf{E}_x, \mathbf{E}_y, \mathbf{E}_z\}$, the position of the collar is given as

$$\mathbf{r} = x\mathbf{E}_x - L\mathbf{E}_y \tag{3.56}$$

Since reference frame $\mathcal{F}$ is fixed and L is constant, the velocity of the collar in reference frame $\mathcal{F}$ is given as

$${}^{\mathcal{F}}\mathbf{v} = \frac{{}^{\mathcal{F}}d\mathbf{r}}{dt} = \dot{x}\mathbf{E}_x \tag{3.57}$$

Furthermore, the acceleration of the collar in reference frame $\mathcal{F}$ is given as

$${}^{\mathcal{F}}\mathbf{a} = \frac{{}^{\mathcal{F}}d}{dt}\left({}^{\mathcal{F}}\mathbf{v}\right) = \ddot{x}\mathbf{E}_x \tag{3.58}$$

Next, using the free body diagram of the collar as shown in Fig. 3-3, we have that

$$\begin{array}{rcl} \mathbf{F}_s & = & \text{Spring Force} \\ \mathbf{N} & = & \text{Reaction Force of Rod on Collar} \end{array}$$

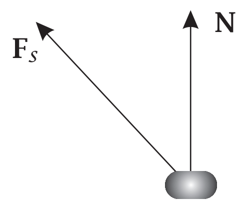


Figure 3-3 Free Body Diagram for Question 3.2

Since the reaction force acts in the $\mathbf{E}_y$ direction, we have that

$$\mathbf{N} = N\mathbf{E}_y \tag{3.59}$$

Next, the force in a linear spring is given as

$$\mathbf{F}_s = -K(\ell - \ell_0)\mathbf{u}_s \tag{3.60}$$

First, the stretched length of the spring is

$$\ell = \|\mathbf{r} - \mathbf{r}_A\| \tag{3.61}$$

where the position of the attachment point is zero, i.e., $\mathbf{r}_A = \mathbf{0}$. Therefore, the stretched length of the spring is given as

$$\ell = \|\mathbf{r}\| = \|x\mathbf{E}_x - L\mathbf{E}_y\| = \sqrt{x^2 + L^2} \tag{3.62}$$

Furthermore, the unstretched length of the spring is given as

$$\ell_0 = L \tag{3.63}$$

Finally, the direction from the attachment point to the particle, $\mathbf{u}_s$, is given as

$$\mathbf{u}_s = \frac{\mathbf{r} - \mathbf{r}_A}{\|\mathbf{r} - \mathbf{r}_A\|} = \frac{x\mathbf{E}_x - L\mathbf{E}_y}{\sqrt{x^2 + L^2}} \tag{3.64}$$

Consequently, the force of the spring is given as

$$\mathbf{F}_s = -K\left[\sqrt{x^2 + L^2} - L\right]\frac{x\mathbf{E}_x - L\mathbf{E}_y}{\sqrt{x^2 + L^2}} \tag{3.65}$$

Grouping this last expression into components, we obtain

$$\mathbf{F}_s = -K\left[\sqrt{x^2 + L^2} - L\right]\frac{x}{\sqrt{x^2 + L^2}}\mathbf{E}_x + K\left[\sqrt{x^2 + L^2} - L\right]\frac{L}{\sqrt{x^2 + L^2}}\mathbf{E}_y \tag{3.66}$$

The resultant force acting on the particle is then given as

$$\mathbf{F}_s = -K\left[\sqrt{x^2 + L^2} - L\right]\frac{x}{\sqrt{x^2 + L^2}}\mathbf{E}_x + \left(N + K\left[\sqrt{x^2 + L^2} - L\right]\frac{L}{\sqrt{x^2 + L^2}}\right)\mathbf{E}_y \tag{3.67}$$

Applying Newton's 2^{nd} Law, we obtain

$$-K\left[\sqrt{x^2+L^2}-L\right]\frac{x}{\sqrt{x^2+L^2}}\mathbf{E}_x+\left(N+K\left[\sqrt{x^2+L^2}-L\right]\frac{L}{\sqrt{x^2+L^2}}\right)\mathbf{E}_y=m\ddot{x}\mathbf{E}_x \tag{3.68}$$

Using the $\mathbf{E}_x$-component of the last equation, we obtain

$$m\ddot{x}=-K\left[\sqrt{x^2+L^2}-L\right]\frac{x}{\sqrt{x^2+L^2}} \tag{3.69}$$

Rearranging this last equation, we obtain the differential equation of motion as

$$m\ddot{x}+K\left[\sqrt{x^2+L^2}-L\right]\frac{x}{\sqrt{x^2+L^2}}=0 \tag{3.70}$$

Solution to Question 3–3

A bead of mass m slides along a fixed circular helix of radius R and constant helical inclination angle ϕ as shown in Fig. P3-3. The equation for the helix is given in cylindrical coordinates as

$$z = R\theta \tan\phi \tag{3.71}$$

Knowing that gravity acts vertically downward, determine the differential equation of motion for the bead in terms of the angle θ using (a) Newton's 2^{nd} law and (b) the work-energy theorem for a particle. In addition assuming the initial conditions $\theta(t=0) = \theta_0$ and $\dot{\theta}(t=0) = \dot{\theta}_0$, determine (c) the displacement attained by the bead when it reaches its maximum height on the helix.

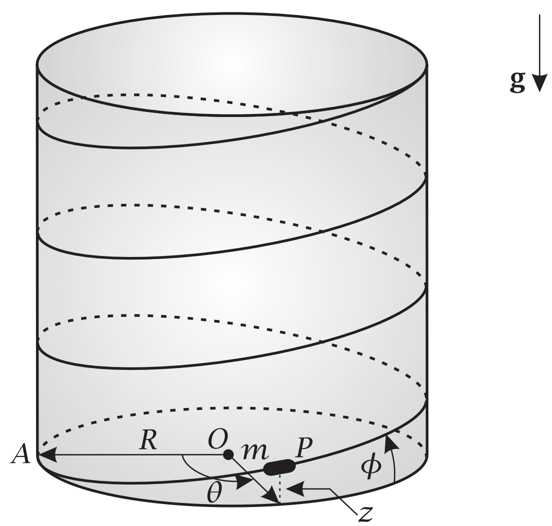


Figure P3-3

Solution to Question 3–3

Kinematics

Let $\mathcal{F}$ be a reference frame fixed to the helix. Then, choose the following coordinate system fixed in reference frame $\mathcal{F}$:

$$\begin{array}{ccc} & \text{Origin at } O & \\ \mathbf{E}_x & = & \text{Along } \mathbf{e}_r \text{ at } t=0 \\ \mathbf{E}_y & = & \text{Along } \mathbf{e}_\theta \text{ at } t=0 \\ \mathbf{E}_z & = & \mathbf{e}_r \times \mathbf{e}_\theta \end{array}$$

Next, let $\mathcal{A}$ be a reference frame that rotates with the projection of the position of particle into the $\{\mathbf{E}_x, \mathbf{E}_y\}$-plane. Corresponding to $\mathcal{A}$, we choose the following coordinate system to describe the motion of the particle:

$$\begin{array}{lcl} & \text{Origin at } O & \\ \mathbf{e}_r & = & \text{Along Radial Direction of Circle} \\ \mathbf{e}_z & = & \mathbf{E}_z \text{ from Reference Frame } \mathcal{F} \\ \mathbf{e}_\theta & = & \mathbf{e}_z \times \mathbf{e}_r \end{array}$$

Now, since ϕ is the angle formed by the helix with the horizontal, we have from the geometry that

$$z = R\theta \tan\phi \tag{3.72}$$

Suppose now that we make the following substitution:

$$\alpha \coloneqq \tan\phi \tag{3.73}$$

Then the position of the bead can be written as

$$\mathbf{r} = R\mathbf{e}_r + \tan\phi R\theta\mathbf{e}_z = R\mathbf{e}_r + \alpha R\theta\mathbf{e}_z \tag{3.74}$$

Furthermore, the angular velocity of reference frame $\mathcal{A}$ in reference frame $\mathcal{F}$ is given as

$${}^{\mathcal{F}}\boldsymbol{\omega}^{\mathcal{A}} = \dot{\theta}\mathbf{e}_z \tag{3.75}$$

Then, differentiating Eq. (3.74) in reference frame $\mathcal{F}$, the velocity of the bead is given as

$${}^{\mathcal{F}}\mathbf{v} = \frac{{}^{\mathcal{F}}d\mathbf{r}}{dt} = \frac{{}^{\mathcal{A}}d\mathbf{r}}{dt} + {}^{\mathcal{F}}\boldsymbol{\omega}^{\mathcal{A}} \times \mathbf{r} \tag{3.76}$$

where

$$\begin{aligned} \frac{{}^{\mathcal{A}}d\mathbf{r}}{dt} &= \alpha R\dot{\theta}\mathbf{e}_z \\ {}^{\mathcal{F}}\boldsymbol{\omega}^{\mathcal{A}} \times \mathbf{r} &= \dot{\theta}\mathbf{e}_z \times (R\mathbf{e}_r + \alpha R\theta\mathbf{e}_z) = R\dot{\theta}\mathbf{e}_\theta \end{aligned} \tag{3.77}$$

Adding the two expressions in Eq. (3.77), we obtain

$${}^{\mathcal{F}}\mathbf{v} = R\dot{\theta}\mathbf{e}_\theta + \alpha R\dot{\theta}\mathbf{e}_z \tag{3.78}$$

The speed in reference frame $\mathcal{F}$ is then given as

$${}^{\mathcal{F}}v = \|{}^{\mathcal{F}}\mathbf{v}\| = R\dot{\theta}\sqrt{1+\alpha^2} \equiv \frac{d}{dt}\left({}^{\mathcal{F}}s\right) \tag{3.79}$$

Consequently,

$${}^{\mathcal{F}}ds = R\sqrt{1+\alpha^2}\,d\theta \tag{3.80}$$

Integrating both sides of Eq. (3.80), we obtain

$$\int_{{}^{\mathcal{F}}s_0}^{{}^{\mathcal{F}}s} ds = \int_{\theta_0}^{\theta} R\sqrt{1+\alpha^2}\,d\theta \tag{3.81}$$

We then obtain

$$ {}^{\mathcal{F}}s - {}^{\mathcal{F}}s_0 = R\sqrt{1+\alpha^2}(\theta - \theta_0) \tag{3.82} $$

Solving Eq. (3.82) for s, the arclength is given as

$$ {}^{\mathcal{F}}s = {}^{\mathcal{F}}s_0 + R\sqrt{1+\alpha^2}(\theta - \theta_0) \tag{3.83} $$

Now the tangent vector in reference frame $\mathcal{F}$ is given as

$$ \mathbf{e}_t = \frac{{}^{\mathcal{F}}\mathbf{v}}{{}^{\mathcal{F}}v} \tag{3.84} $$

Using the speed from Eq. (3.79) and the velocity from Eq. (3.78), the tangent vector in reference frame $\mathcal{F}$ is obtained as Substituting the expressions for ${}^{\mathcal{F}}\mathbf{v}$ and ${}^{\mathcal{F}}v$ from part (a) into Eq. (3.84), we obtain

$$ \mathbf{e}_t = \frac{R\dot{\theta}\mathbf{e}_\theta + \alpha R\dot{\theta}\mathbf{e}_z}{R\dot{\theta}\sqrt{1+\alpha^2}} \tag{3.85} $$

Simplifying Eq. (3.85), we have

$$ \mathbf{e}_t = \frac{\mathbf{e}_\theta + \alpha\mathbf{e}_z}{\sqrt{1+\alpha^2}} \tag{3.86} $$

Next, we have

$$ \frac{{}^{\mathcal{F}}d\mathbf{e}_t}{dt} = \kappa\, {}^{\mathcal{F}}v\,\mathbf{e}_n \tag{3.87} $$

Applying the rate of change transport theorem between reference frames $\mathcal{A}$ and $\mathcal{F}$, we have

$$ \frac{{}^{\mathcal{F}}d\mathbf{e}_t}{dt} = \frac{{}^{\mathcal{A}}d\mathbf{e}_t}{dt} + {}^{\mathcal{F}}\boldsymbol{\omega}^{\mathcal{A}} \times \mathbf{e}_t \tag{3.88} $$

where

$$ \frac{{}^{\mathcal{A}}d\mathbf{e}_t}{dt} = \mathbf{0} \tag{3.89} $$

$$ {}^{\mathcal{F}}\boldsymbol{\omega}^{\mathcal{A}} \times \mathbf{e}_t = \dot{\theta}\mathbf{e}_z \times \frac{\mathbf{e}_\theta + \alpha\mathbf{e}_z}{\sqrt{1+\alpha^2}} = -\frac{\dot{\theta}}{\sqrt{1+\alpha^2}}\mathbf{e}_r \tag{3.90} $$

Adding Eqs. (3.89) and (3.90) gives

$$ \frac{{}^{\mathcal{F}}d\mathbf{e}_t}{dt} = -\frac{\dot{\theta}}{\sqrt{1+\alpha^2}}\mathbf{e}_r \tag{3.91} $$

The principal unit normal is then given as

$$ \mathbf{e}_n = \frac{{}^{\mathcal{F}}d\mathbf{e}_t/dt}{\|{}^{\mathcal{F}}d\mathbf{e}_t/dt\|} = -\mathbf{e}_r \tag{3.92} $$

We then obtain the principal unit bi-normal vector as

$$\mathbf{e}_b = \mathbf{e}_t \times \mathbf{e}_n = \frac{\mathbf{e}_\theta + \alpha\mathbf{e}_z}{\sqrt{1+\alpha^2}} \times (-\mathbf{e}_r) = -\frac{\alpha\mathbf{e}_\theta - \mathbf{e}_z}{\sqrt{1+\alpha^2}} \tag{3.93}$$

Furthermore, the curvature is given as

$$\kappa = \frac{{}^{\mathcal{F}}d\mathbf{e}_t/dt}{{}^{\mathcal{F}}v} = \frac{1}{R(1+\alpha^2)} \tag{3.94}$$

The acceleration in reference frame $\mathcal{F}$ is then given as

$$^{\mathcal{F}}\mathbf{a} = \frac{d}{dt}\left({}^{\mathcal{F}}v\right)\mathbf{e}_t + \kappa\left({}^{\mathcal{F}}v\right)^2\mathbf{e}_n \tag{3.95}$$

Using the expression for ${}^{\mathcal{F}}v$ from Eq. (3.79) we have that

$$\frac{d}{dt}\left({}^{\mathcal{F}}v\right) = R\ddot{\theta}\sqrt{1+\alpha^2} \tag{3.96}$$

Also, using the curvature from Eq. (3.94), we have that

$$\kappa\left({}^{\mathcal{F}}v\right)^2 = \frac{1}{R(1+\alpha^2)}\left[R\dot{\theta}\sqrt{1+\alpha^2}\right]^2 = R\dot{\theta}^2 \tag{3.97}$$

Thus, the acceleration is given as

$$^{\mathcal{F}}\mathbf{a} = R\ddot{\theta}\sqrt{1+\alpha^2}\mathbf{e}_t + R\dot{\theta}^2\mathbf{e}_n \tag{3.98}$$

Kinetics

Using the free body diagram in Fig. 3-4, we have that

$\mathbf{N}_n$	=	Reaction Force of Track on Bead in $\mathbf{e}_n$ Direction
$\mathbf{N}_b$	=	Reaction Force of Track on Bead in $\mathbf{e}_b$ Direction
$m\mathbf{g}$	=	Force of Gravity

Therefore,

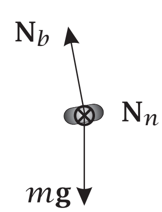


Figure 3-4 Free Body Diagram for Question 3.4

$$\mathbf{F} = \mathbf{N}_n + \mathbf{N}_b + m\mathbf{g} \tag{3.99}$$

From the geometry we have that

$$\begin{aligned} \mathbf{N}_n &= N_n \mathbf{e}_n && (3.100)\\ \mathbf{N}_b &= N_b \mathbf{e}_n && (3.101)\\ m\mathbf{g} &= -mg\mathbf{e}_z && (3.102) \end{aligned}$$

Consequently,

$$\mathbf{F} = N_n \mathbf{e}_n + N_b \mathbf{e}_b - mg\mathbf{e}_z \tag{3.103}$$

Now from Eqs. (3.86) and (3.93) we have

$$\begin{aligned} \mathbf{e}_t &= \frac{\mathbf{e}_\theta + \alpha \mathbf{e}_z}{\sqrt{1+\alpha^2}} \\ \mathbf{e}_b &= -\frac{\alpha \mathbf{e}_\theta - \mathbf{e}_z}{\sqrt{1+\alpha^2}} \end{aligned} \tag{3.104}$$

Using Eq. (3.104) we can obtain an expression for $\mathbf{e}_z$ in terms of $\mathbf{e}_t$ and $\mathbf{e}_b$. First, multiplying $\mathbf{e}_t$ by $\alpha\sqrt{1+\alpha^2}$ and multiplying $\mathbf{e}_b$ by $-\sqrt{1+\alpha^2}$, we obtain

$$\begin{aligned} \alpha\sqrt{1+\alpha^2}\mathbf{e}_t &= \alpha \mathbf{e}_\theta + \alpha^2 \mathbf{e}_z \\ -\sqrt{1+\alpha^2}\mathbf{e}_b &= \alpha \mathbf{e}_\theta - \mathbf{e}_z \end{aligned} \tag{3.105}$$

Subtracting these last two equations gives

$$\alpha\sqrt{1+\alpha^2}\mathbf{e}_t + \sqrt{1+\alpha^2}\mathbf{e}_b = (1+\alpha^2)\mathbf{e}_z \tag{3.106}$$

Solving this last equation for $\mathbf{e}_z$, we obtain

$$\mathbf{e}_z = \frac{\alpha \mathbf{e}_t + \mathbf{e}_b}{\sqrt{1+\alpha^2}} \tag{3.107}$$

The force $\mathbf{F}$ given in Eq. (3.103) can then be written as

$$\mathbf{F} = N_n \mathbf{e}_n + N_b \mathbf{e}_b - mg\left[\frac{\alpha \mathbf{e}_t + \mathbf{e}_b}{\sqrt{1+\alpha^2}}\right] \tag{3.108}$$

Separating this last equation into components, we obtain

$$\mathbf{F} = -\frac{mg\alpha}{\sqrt{1+\alpha^2}}\mathbf{e}_t + N_n \mathbf{e}_n + \left(N_b - \frac{mg}{\sqrt{1+\alpha^2}}\right)\mathbf{e}_b \tag{3.109}$$

(a) Differential Equation Using Newton's 2^{nd} Law

Setting $\mathbf{F} = m\,{}^{\mathcal{F}}\mathbf{a}$ using the expression for ${}^{\mathcal{F}}\mathbf{a}$ from Eq. (3.98), we obtain

$$-\frac{mg\alpha}{\sqrt{1+\alpha^2}}\mathbf{e}_t + N_n \mathbf{e}_n + \left(N_b - \frac{mg}{\sqrt{1+\alpha^2}}\right)\mathbf{e}_b = mR\ddot{\theta}\sqrt{1+\alpha^2}\mathbf{e}_t + mR\dot{\theta}^2\mathbf{e}_n \tag{3.110}$$

Equating components in Eq. (3.110) yields the following three scalar equations:

$$-\frac{mg\alpha}{\sqrt{1+\alpha^2}} = mR\ddot{\theta}\sqrt{1+\alpha^2} \tag{3.111}$$

$$N_n = mR\dot{\theta}^2 \tag{3.112}$$

$$N_b = \frac{mg}{\sqrt{1+\alpha^2}} \tag{3.113}$$

It is noted that, because it contains no reaction forces, Eq. (3.111) is the differential equation of motion for the particle, i.e., the differential equation of motion is given as

$$mR\ddot{\theta}\sqrt{1+\alpha^2} + \frac{mg\alpha}{\sqrt{1+\alpha^2}} = 0 \tag{3.114}$$

Eq. (3.114) can be rewritten as

$$mR(1+\alpha^2)\ddot{\theta} + mg\alpha = 0 \tag{3.115}$$

Simplifying Eq. (3.115), we obtain

$$\ddot{\theta} + \frac{g}{R(1+\alpha^2)}\alpha = 0 \tag{3.116}$$

Then, using the the fact that $\alpha = \tan\phi$ from Eq. (3.73), we obtain

$$\ddot{\theta} + \frac{g}{R(1+\tan^2\phi)}\tan\phi = 0 \tag{3.117}$$

Now from trigonometry we have that

$$1 + \tan^2\phi = \sec^2\phi \tag{3.118}$$

Using the result of Eq. (3.118) in Eq. (3.117), we obtain the differential equation of motion as

$$\ddot{\theta} + \frac{g\tan\phi}{R\sec^2\phi} = 0 \tag{3.119}$$

(b) Differential Equation Using Work-Energy Theorem

Applying the work-energy theorem to the bead, we have

$$\frac{d}{dt}\left({}^{\mathcal{F}}T\right) = \mathbf{F}\cdot{}^{\mathcal{F}}\mathbf{v} \tag{3.120}$$

Using the expression for ${}^{\mathcal{F}}\mathbf{v}$ from Eq. (3.78), the kinetic energy in reference frame $\mathcal{F}$ is given as

$${}^{\mathcal{F}}T = \frac{1}{2}m{}^{\mathcal{F}}\mathbf{v}\cdot{}^{\mathcal{F}}\mathbf{v} = \frac{1}{2}m(R^2\dot{\theta}^2\alpha^2R^2\dot{\theta}^2) = \frac{1}{2}mR^2(1+\alpha^2)\dot{\theta}^2 \tag{3.121}$$

Computing the rate of change of kinetic energy, we obtain

$$\frac{d}{dt}\left({}^{\mathcal{F}}T\right) = mR^2(1+\alpha^2)\dot{\theta}\ddot{\theta} \tag{3.122}$$

Next, using the resultant force acting on the bead as given in Eq. (3.109), the power produced by all forces is given as

$$\mathbf{F}\cdot{}^{\mathcal{F}}\mathbf{v} = \left(-\frac{mg\alpha}{\sqrt{1+\alpha^2}}\mathbf{e}_t + N_n\mathbf{e}_n + \left(N_b - \frac{mg}{\sqrt{1+\alpha^2}}\right)\mathbf{e}_b\right)\cdot{}^{\mathcal{F}}\mathbf{v} \tag{3.123}$$

Recalling by definition that ${}^{\mathcal{F}}\mathbf{v} = {}^{\mathcal{F}}v\mathbf{e}_t$, Eq. (3.123) simplifies to

$$\mathbf{F}\cdot{}^{\mathcal{F}}\mathbf{v} = -\frac{mg\alpha}{\sqrt{1+\alpha^2}}{}^{\mathcal{F}}v \tag{3.124}$$

Then, substituting the expression for ${}^{\mathcal{F}}v$ from Eq. (3.79), we have

$$\mathbf{F}\cdot{}^{\mathcal{F}}\mathbf{v} = -\frac{mg\alpha}{\sqrt{1+\alpha^2}}R\dot{\theta}\sqrt{1+\alpha^2} \tag{3.125}$$

Setting Eq. (3.122) equal to Eq. (3.125), we obtain

$$mR^2(1+\alpha^2)\dot{\theta}\ddot{\theta} = -\frac{mg\alpha}{\sqrt{1+\alpha^2}}R\dot{\theta}\sqrt{1+\alpha^2} \tag{3.126}$$

Rearranging Eq. (3.126) yields

$$\dot{\theta}\left[mR^2(1+\alpha^2)\ddot{\theta} + \frac{mg\alpha}{\sqrt{1+\alpha^2}}R\sqrt{1+\alpha^2}\right] = 0 \tag{3.127}$$

Observing that $\dot{\theta} \neq 0$ as a function of time, the differential equation of motion is obtained as

$$mR^2(1+\alpha^2)\ddot{\theta} + \frac{mg\alpha}{\sqrt{1+\alpha^2}}R\sqrt{1+\alpha^2} = 0 \tag{3.128}$$

(c) Maximum Displacement of Bead

For this particular problem, we can obtain the maximum distance traveled using the alternate form of the work-energy theorm for a particle. In particular, we know that

$$\frac{d}{dt}\left({}^{\mathcal{F}}E\right) = \mathbf{F}^{nc}\cdot{}^{\mathcal{F}}\mathbf{v} \tag{3.129}$$

Now since the force of gravity is conservative, we know that the only possible non-conservative forces are due to the reaction of the track on the bead, i.e.,

$$\mathbf{F}^{nc} = \mathbf{N}_n + \mathbf{N}_b \tag{3.130}$$

Using the expressions for $\mathbf{N}_n$ and $\mathbf{N}_b$ from Eq. (3.100) and Eq. (3.101), we have that

$$\mathbf{F}^{nc} = N_n\mathbf{e}_n + N_b\mathbf{e}_b \tag{3.131}$$

Furthermore, since $\mathbf{e}_n$ and $\mathbf{e}_b$ both lie in the direction orthogonal to $\mathbf{e}_t$, we have that

$$\begin{aligned} \mathbf{e}_n \cdot \mathbf{e}_t &= 0 \\ \mathbf{e}_b \cdot \mathbf{e}_t &= 0 \end{aligned} \tag{3.132}$$

Furthermore, since ${}^{\mathcal{F}}\mathbf{v} = {}^{\mathcal{F}}v\mathbf{e}_t$, we know that

$$\mathbf{F}^{nc} = (N_n\mathbf{e}_n + N_b\mathbf{e}_b) \cdot {}^{\mathcal{F}}v\mathbf{e}_t = 0 \tag{3.133}$$

Consequently,

$$\frac{d}{dt}\left({}^{\mathcal{F}}E\right) = 0 \tag{3.134}$$

Integrating Eq. (3.134), we obtain

$${}^{\mathcal{F}}E = \text{constant} \tag{3.135}$$

Now since ${}^{\mathcal{F}}E = {}^{\mathcal{F}}T + {}^{\mathcal{F}}U$, we have

$${}^{\mathcal{F}}T + {}^{\mathcal{F}}U = \text{constant} \tag{3.136}$$

Next, we know that the bead will attain its maximum distance when its velocity is zero, i.e., the maximum distance will be attained when $\dot{\theta} = 0$. Using Eq. (3.136), we have that

$${}^{\mathcal{F}}T_0 + {}^{\mathcal{F}}U_0 = {}^{\mathcal{F}}T_1 + {}^{\mathcal{F}}U_1 \tag{3.137}$$

where the subscript "0" is at time $t = t_0 = 0$, and the subscript "1" is at time $t = t_1$ when $\dot{\theta} = 0$. We already have the kinetic energy of the bead from Eq. (3.121). Next, since the only conservative force acting on the bead is due to gravity, the potential energy of the bead in reference frame $\mathcal{F}$ is given as

$${}^{\mathcal{F}}U = {}^{\mathcal{F}}U_g = -m\mathbf{g} \cdot \mathbf{r} \tag{3.138}$$

Substituting the expression for $\mathbf{r}$ from Eq. (3.74) and the expression for $m\mathbf{g}$ from Eq. (3.102) into Eq. (3.138), we obtain

$${}^{\mathcal{F}}U = {}^{\mathcal{F}}U_g = mg\mathbf{e}_z \cdot (R\mathbf{e}_r + \alpha R\theta\mathbf{e}_z) = mgR\theta\alpha \tag{3.139}$$

Then, substituting the expression for ${}^{\mathcal{F}}T$ and ${}^{\mathcal{F}}U$ from Eqs. (3.121) and 3.139 into Eq. (3.136), we obtain

$$\frac{1}{2}mR^2\dot{\theta}^2(1+\alpha^2) + mgR\theta\alpha = \text{constant} \tag{3.140}$$

Furthermore, applying Eq. (3.137), we obtain

$$\frac{1}{2}mR^2\dot{\theta}_0^2(1+\alpha^2) + mgR\theta_0\alpha = \frac{1}{2}mR^2\dot{\theta}_1^2(1+\alpha^2) + mgR\theta_1\alpha \tag{3.141}$$

Now we know that since the maximum distance is obtained when the velocity of the bead is zero, we must have that $\dot{\theta}_1 = 0$. Furthermore, since the initial value of θ is zero, we have that $\theta_0 = 0$. Consequently, Eq. (3.141) reduces to

$$\frac{1}{2}mR^2\dot{\theta}_0^2(1+\alpha^2) = mgR\theta_1\alpha \qquad (3.142)$$

Solving Eq. (3.142) for θ_1, we obtain

$$\theta_1 = \frac{R\dot{\theta}_0^2(1+\alpha^2)}{2g\alpha} \qquad (3.143)$$

Finally, since the distance traveled along the helix is equivalent to the arclength, the distance traveled along the helix is given from Eq. (3.83) as

$$\mathcal{F}_s = R\sqrt{1+\alpha^2}\frac{R\dot{\theta}_0^2(1+\alpha^2)}{2g\alpha} \qquad (3.144)$$

Simplifying Eq. (3.144), we obtain the maximum distance traveled along the incline as

$$\mathcal{F}_s = \frac{R^2\dot{\theta}_0^2}{2g\alpha}\left(1+\alpha^2\right)^{3/2} \qquad (3.145)$$

Question 3–5

A collar of mass m is constrained to move along a frictionless track in the form of a logarithmic spiral as shown in Fig. P3-5. The equation for the spiral is given as

$$r = r_0 e^{-a\theta}$$

where r_0 and a are constants and θ is the angle as shown in the figure. Assuming that gravity acts downward, determine the differential equation of motion in terms of the angle θ using (a) Newton's 2^{nd} law and (b) the work-energy theorem for a particle.

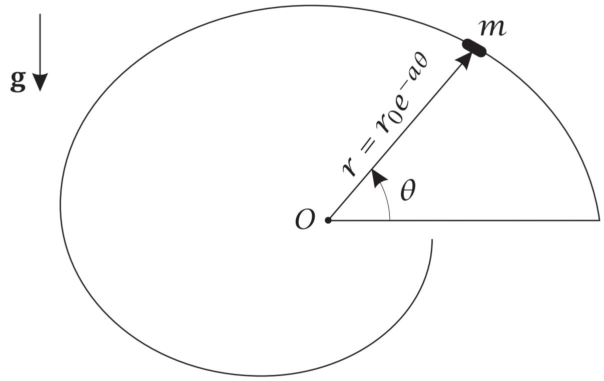


Figure P3-5

Solution to Question 3–5

Kinematics

Let $\mathcal{F}$ be a reference frame fixed to the track. Then, choose the following coordinate system fixed in reference frame $\mathcal{F}$:

Origin at O

$\mathbf{E}_x$	=	To the Right
$\mathbf{E}_z$	=	Out of Page
$\mathbf{E}_y$	=	$\mathbf{E}_z \times \mathbf{E}_x$

Next, let $\mathcal{A}$ be a reference frame fixed to the direction of Om. Then, we choose the following coordinate system fixed in reference frame $\mathcal{F}$:

Origin at O

$\mathbf{e}_r$	=	Along Om
$\mathbf{E}_z$	=	Out of Page
$\mathbf{e}_\theta$	=	$\mathbf{E}_z \times \mathbf{e}_r$

The relationship between the bases $\{\mathbf{E}_x, \mathbf{E}_y, \mathbf{E}_z\}$ and $\{\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_z\}$ is given as

$$\begin{aligned} \mathbf{e}_r &= \cos\theta\mathbf{E}_x + \sin\theta\mathbf{E}_y &(3.146)\\ \mathbf{e}_\theta &= -\sin\theta\mathbf{E}_x + \cos\theta\mathbf{E}_y &(3.147)\end{aligned}$$

The position of the particle is then given as

$$\mathbf{r} = r\mathbf{e}_r = r_0 e^{-a\theta}\mathbf{e}_r \tag{3.148}$$

Furthermore, the angular velocity of reference frame $\mathcal{A}$ in reference frame $\mathcal{F}$ is given as

$${}^{\mathcal{F}}\boldsymbol{\omega}^{\mathcal{A}} = \dot\theta\mathbf{E}_z \tag{3.149}$$

Applying the rate of change transport theorem between reference frames $\mathcal{A}$ and $\mathcal{F}$, we obtain the velocity of the particle in reference frame $\mathcal{F}$ as

$${}^{\mathcal{F}}\mathbf{v} = \frac{{}^{\mathcal{F}}d\mathbf{r}}{dt} = \frac{{}^{\mathcal{A}}d\mathbf{r}}{dt} + {}^{\mathcal{F}}\boldsymbol{\omega}^{\mathcal{A}} \times \mathbf{r} \tag{3.150}$$

where

$$\begin{aligned} \frac{{}^{\mathcal{A}}d\mathbf{r}}{dt} &= \dot r\mathbf{e}_r = -ar_0\dot\theta e^{-a\theta}\mathbf{e}_r &(3.151)\\ {}^{\mathcal{F}}\boldsymbol{\omega}^{\mathcal{A}} \times \mathbf{r} &= \dot\theta\mathbf{E}_z \times r\mathbf{e}_r = \dot\theta\mathbf{E}_z \times r_0 e^{-a\theta}\mathbf{e}_r = r_0\dot\theta e^{-a\theta}\mathbf{e}_\theta &(3.152)\end{aligned}$$

Adding the expressions in Eq. (3.151) and Eq. (3.152), we obtain the velocity of the particle in reference frame $\mathcal{F}$ as

$${}^{\mathcal{F}}\mathbf{v} = -ar_0\dot\theta e^{-a\theta}\mathbf{e}_r + r_0\dot\theta e^{-a\theta}\mathbf{e}_\theta \tag{3.153}$$

Simplifying Eq. (3.150), we obtain ${}^{\mathcal{F}}\mathbf{v}$ as

$${}^{\mathcal{F}}\mathbf{v} = r_0\dot\theta e^{-a\theta}\left[-a\mathbf{e}_r + \mathbf{e}_\theta\right] \tag{3.154}$$

Now we need the acceleration of the collar in reference frame $\mathcal{F}$. For this problem it is most convenient to obtain ${}^{\mathcal{F}}\mathbf{a}$ in terms of an intrinsic basis as viewed by an observer fixed to the track. First, the tangent vector is given as

$$\mathbf{e}_t = \frac{{}^{\mathcal{F}}\mathbf{v}}{\|{}^{\mathcal{F}}\mathbf{v}\|} = \frac{{}^{\mathcal{F}}\mathbf{v}}{\|{}^{\mathcal{F}}v\|} \tag{3.155}$$

where ${}^{\mathcal{F}}v$ is the speed of the particle in reference frame $\mathcal{F}$. Now we know that the speed of the particle in reference frame $\mathcal{F}$ is given as

$${}^{\mathcal{F}}v = \|{}^{\mathcal{F}}\mathbf{v}\| = r_0\dot\theta e^{-a\theta}\sqrt{a^2+1} \tag{3.156}$$

Dividing ${}^{\mathcal{F}}\mathbf{v}$ in Eq. (3.154) by ${}^{\mathcal{F}}v$ in Eq. (3.156), we obtain $\mathbf{e}_t$ as

$$\mathbf{e}_t = \frac{-a\mathbf{e}_r + \mathbf{e}_\theta}{\sqrt{a^2+1}} \tag{3.157}$$

Then, the principle unit normal vector is obtained as

$$\mathbf{e}_n = \frac{{}^{\mathcal{F}} d\mathbf{e}_t/dt}{\|{}^{\mathcal{F}} d\mathbf{e}_t/dt\|} \tag{3.158}$$

Now we have from the basic kinematic equation that

$$\frac{{}^{\mathcal{F}}d\mathbf{e}_t}{dt} = \frac{{}^{\mathcal{A}}d\mathbf{e}_t}{dt} + {}^{\mathcal{F}}\boldsymbol{\omega}^{\mathcal{A}} \times \mathbf{e}_t \tag{3.159}$$

where

$$\begin{aligned} \frac{{}^{\mathcal{F}}d\mathbf{e}_t}{dt} &= \mathbf{0} \\ {}^{\mathcal{F}}\boldsymbol{\omega}^{\mathcal{A}} \times \mathbf{e}_t &= \dot{\theta}\mathbf{E}_z \times \frac{-a\mathbf{e}_r + \mathbf{e}_\theta}{\sqrt{a^2+1}} = -\dot{\theta}\frac{\mathbf{e}_r + a\mathbf{e}_\theta}{\sqrt{a^2+1}} \end{aligned} \tag{3.160}$$

Adding the expressions in Eq. (3.160), we obtain

$$\frac{{}^{\mathcal{F}}d\mathbf{e}_t}{dt} = -\dot{\theta}\frac{\mathbf{e}_r + a\mathbf{e}_\theta}{\sqrt{a^2+1}} \tag{3.161}$$

Consequently,

$$\left\|\frac{{}^{\mathcal{F}}d\mathbf{e}_t}{dt}\right\| = \dot{\theta} \tag{3.162}$$

Dividing ${}^{\mathcal{F}} d\mathbf{e}_t/dt$ in Eq. (3.161) by $\|{}^{\mathcal{F}} d\mathbf{e}_t/dt\|$ in Eq. (3.162), we obtain $\mathbf{e}_n$ as

$$\mathbf{e}_n = -\frac{\mathbf{e}_r + a\mathbf{e}_\theta}{\sqrt{a^2+1}} \tag{3.163}$$

Furthermore, the curvature, κ, is obtained as

$$\kappa = \frac{\|{}^{\mathcal{F}} d\mathbf{e}_t/dt\|}{{}^{\mathcal{F}}v} \tag{3.164}$$

Substituting $\|{}^{\mathcal{F}} d\mathbf{e}_t/dt\|$ from Eq. (3.162) and ${}^{\mathcal{F}}v$ from Eq. (3.156), we obtain κ as

$$\kappa = \frac{1}{r_0 e^{-a\theta}\sqrt{a^2+1}} \tag{3.165}$$

The acceleration is then given in terms of the intrinsic basis as

$${}^{\mathcal{F}}\mathbf{a} = \frac{d}{dt}\left({}^{\mathcal{F}}v\right)\mathbf{e}_t + \kappa\left({}^{\mathcal{F}}v\right)^2\mathbf{e}_n \tag{3.166}$$

Now we have that

$$\frac{d}{dt}\left({}^{\mathcal{F}}v\right) = r_0(\ddot{\theta} - a\dot{\theta}^2)e^{-a\theta}\sqrt{a^2+1} \tag{3.167}$$

Furthermore,

$$\kappa\left({}^{\mathcal{F}}v\right)^2 = \frac{1}{r_0 e^{-a\theta}\sqrt{a^2+1}} r_0^2\dot{\theta}^2 e^{-2a\theta}(a^2+1) = r_0\dot{\theta}^2 e^{-a\theta}\sqrt{a^2+1} \tag{3.168}$$

The acceleration in reference frame $\mathcal{F}$ is then given as

$${}^{\mathcal{F}}\mathbf{a} = r_0 e^{-a\theta}\sqrt{a^2+1}(\ddot{\theta} - a\dot{\theta}^2)\mathbf{e}_t + r_0\dot{\theta}^2 e^{-a\theta}\sqrt{a^2+1}\mathbf{e}_n \tag{3.169}$$

Kinetics

The free body diagram of the particle is shown in Fig. 3-5. It can be seen that

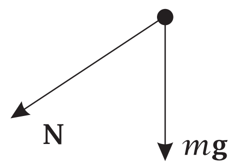


Figure 3-5 Free Body Diagram fof Question 3–5.

the following two forces act on the collar: (1) the reaction force of the track, $\mathbf{N}$, and (2) gravity, $m\mathbf{g}$. Since $\mathbf{N}$ acts in the direction normal to the track, we have

$$\mathbf{N} = N_n \mathbf{e}_n \tag{3.170}$$

Now, since gravity acts vertically downward, we have that

$$m\mathbf{g} = -mg\mathbf{E}_y \tag{3.171}$$

where $\mathbf{E}_y$ is the vertically upward direction. Resolving $\mathbf{E}_y$ in the basis $\{\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{E}_z\}$, we have that

$$\mathbf{E}_y = \sin\theta\,\mathbf{e}_r + \cos\theta\,\mathbf{e}_\theta \tag{3.172}$$

Consequently, the force of gravity can be written as

$$m\mathbf{g} = -mg\sin\theta\,\mathbf{e}_r - mg\cos\theta\,\mathbf{e}_\theta \tag{3.173}$$

Then the total force acting on the collar is given as

$$\mathbf{F} = \mathbf{N} + m\mathbf{g} = N_n\mathbf{e}_n - mg\sin\theta\,\mathbf{e}_r - mg\cos\theta\,\mathbf{e}_\theta \tag{3.174}$$

(a) Differential Equation Using Newton's 2^{nd} Law

Setting $\mathbf{F}$ equal to $m\,{}^{\mathcal{F}}\mathbf{a}$ using ${}^{\mathcal{F}}\mathbf{a}$ from Eq. (3.169), we obtain

$$\begin{aligned} N_n\mathbf{e}_n - mg\sin\theta\,\mathbf{e}_r - mg\cos\theta\,\mathbf{e}_\theta &= mr_0e^{-a\theta}\sqrt{a^2+1}(\ddot{\theta} - a\dot{\theta}^2)\mathbf{e}_t \\ &+ mr_0\dot{\theta}^2e^{-a\theta}\sqrt{a^2+1}\mathbf{e}_n \end{aligned} \tag{3.175}$$

Now we know that, in order to obtain the differential equation of motion, we need to eliminate the reaction force exerted by the track. An easy way to eliminate $\mathbf{N}$ is to take the scalar product in the $\mathbf{e}_t$-direction on both sides of Eq. (3.175). Noting that $\mathbf{e}_n \cdot \mathbf{e}_t = 0$, we then obtain

$$-mg\sin\theta\,\mathbf{e}_r\cdot\mathbf{e}_t - mg\cos\theta\,\mathbf{e}_\theta\cdot\mathbf{e}_t = mr_0e^{-a\theta}\sqrt{a^2+1}(\ddot{\theta} - a\dot{\theta}^2) \tag{3.176}$$

Now, using the expression for $\mathbf{e}_t$ from Eq. (3.157) and the expression for $\mathbf{e}_n$ from Eq. (3.163), we have that

$$\begin{aligned} \mathbf{e}_r \cdot \mathbf{e}_t &= \mathbf{e}_r \cdot \frac{-a\mathbf{e}_r + \mathbf{e}_\theta}{\sqrt{a^2+1}} = -\frac{a}{\sqrt{a^2+1}} \\ \mathbf{e}_\theta \cdot \mathbf{e}_t &= \mathbf{e}_\theta \cdot \frac{-a\mathbf{e}_r + \mathbf{e}_\theta}{\sqrt{a^2+1}} = \frac{1}{\sqrt{a^2+1}} \end{aligned} \tag{3.177}$$

Substituting the expressions in Eq. (3.177) into Eq. (3.176), we obtain

$$mg\sin\theta\frac{a}{\sqrt{a^2+1}} - mg\cos\theta\frac{1}{\sqrt{a^2+1}} = mr_0e^{-a\theta}\sqrt{a^2+1}(\ddot{\theta} - a\dot{\theta}^2) \tag{3.178}$$

Rearranging and simplifying Eq. (3.178), we obtain the differential equation of motion as

$$\ddot{\theta} - a\dot{\theta}^2 + \frac{g}{r_0(a^2+1)}e^{a\theta}(\cos\theta - a\sin\theta) = 0 \tag{3.179}$$

(b) Differential Equation Using Work-Energy Theorem for a Particle

To obtain the differential equation of motion using the work-energy, we choose to apply the *alternate form* of the work-energy theorem for a particle. The alternate form of the work-energy theorem is given in reference frame $\mathcal{F}$ as

$$\frac{d}{dt}\left({}^{\mathcal{F}}E\right) = \mathbf{F}^{nc} \cdot {}^{\mathcal{F}}\mathbf{v} \tag{3.180}$$

Now for this problem we know that the only two forces acting on the particle are the force of gravity and the reaction force of the track. Moreover, we know that the force of gravity is conservative while the reaction force is non-conservative. Therefore, we have $\mathbf{F}^{nc}$ as

$$\mathbf{F}^{nc} = \mathbf{N} \tag{3.181}$$

Now, since $\mathbf{N}$ acts in the direction of $\mathbf{e}_n$ and ${}^{\mathcal{F}}\mathbf{v}$ acts in the direction of $\mathbf{e}_t$, we have that

$$\mathbf{F}^{nc} \cdot {}^{\mathcal{F}}\mathbf{v} = \mathbf{N} \cdot {}^{\mathcal{F}}\mathbf{v} = N\mathbf{e}_n \cdot {}^{\mathcal{F}}v\mathbf{e}_t = 0 \tag{3.182}$$

Consequently,

$$\frac{d}{dt}\left({}^{\mathcal{F}}E\right) = 0 \tag{3.183}$$

Now the total energy in reference frame $\mathcal{F}$ is given as

$${}^{\mathcal{F}}E = {}^{\mathcal{F}}T + {}^{\mathcal{F}}U \tag{3.184}$$

First, the kinetic in reference frame $\mathcal{F}$ is given as

$${}^{\mathcal{F}}T = \frac{1}{2}m{}^{\mathcal{F}}\mathbf{v} \cdot {}^{\mathcal{F}}\mathbf{v} \tag{3.185}$$

Using the expression for ${}^{\mathcal{F}}\mathbf{v}$ from Eq. (3.154), we obtain the kinetic energy in reference frame $\mathcal{F}$ as

$$\begin{aligned} {}^{\mathcal{F}}T &= \frac{1}{2}m\left(r_0\dot{\theta}e^{-a\theta}\left[-a\mathbf{e}_r + \mathbf{e}_\theta\right]\right)\cdot\left(r_0\dot{\theta}e^{-a\theta}\left[-a\mathbf{e}_r + \mathbf{e}_\theta\right]\right) \\ &= \frac{1}{2}mr_0^2(a^2+1)\dot{\theta}^2e^{-2a\theta} \end{aligned} \tag{3.186}$$

Next, since gravity is the only conservative force acting on the particle, the potential energy in reference frame $\mathcal{F}$ is given as

$${}^{\mathcal{F}}U = {}^{\mathcal{F}}U_g \tag{3.187}$$

Now since gravity is a constant force, we have that

$${}^{\mathcal{F}}U_g = -m\mathbf{g}\cdot\mathbf{r} \tag{3.188}$$

Using the expression for $\mathbf{r}$ from Eq. (3.148) and the expression for $m\mathbf{g}$ from Eq. (3.171), we obtain ${}^{\mathcal{F}}U_g$ as

$${}^{\mathcal{F}}U_g = -m\mathbf{g}\cdot\mathbf{r} = -(-mg\mathbf{E}_y)\cdot r_0e^{-a\theta}\mathbf{e}_r = mgr_0e^{-a\theta}\mathbf{E}_y\cdot\mathbf{e}_r \tag{3.189}$$

Using the expression for $\mathbf{e}_r$ from Eq. (3.146), we have that

$$\mathbf{E}_y\cdot\mathbf{e}_r = \mathbf{E}_y\cdot(\cos\theta\mathbf{E}_x + \sin\theta\mathbf{E}_y) = \sin\theta \tag{3.190}$$

Consequently, ${}^{\mathcal{F}}U_g$ can be written as

$${}^{\mathcal{F}}U_g = mgr_0e^{-a\theta}\sin\theta \tag{3.191}$$

Then, adding Eq. (3.186) and Eq. (3.191), the total energy in reference frame $\mathcal{F}$ is given as

$${}^{\mathcal{F}}E = {}^{\mathcal{F}}T + {}^{\mathcal{F}}U = \frac{1}{2}mr_0^2(a^2+1)\dot{\theta}^2e^{-2a\theta} + mgr_0e^{-a\theta}\sin\theta \tag{3.192}$$

Then, computing the rate of change of ${}^{\mathcal{F}}E$, we obtain

$$\begin{aligned} \frac{d}{dt}\left({}^{\mathcal{F}}E\right) &= mr_0^2(a^2+1)\left[\dot{\theta}\ddot{\theta}e^{-2a\theta} = a\dot{\theta}^2\dot{\theta}e^{-2a\theta}\right] \\ &\quad + mgr_0\left[-a\dot{\theta}e^{-a\theta}\sin\theta + \dot{\theta}e^{-a\theta}\cos\theta\right] \end{aligned} \tag{3.193}$$

Eq. (3.193) can be re-written as

$$\frac{d}{dt}\left({}^{\mathcal{F}}E\right) = mr_0^2(a^2+1)\dot{\theta}e^{-2a\theta}\left[\ddot{\theta} - a\dot{\theta}^2\right] + mgr_0\dot{\theta}e^{-a\theta}(-a\sin\theta + \cos\theta) \tag{3.194}$$

Simplifying Eq. (3.194) and setting the result equal to zero, we obtain

$$\frac{d}{dt}\left({}^{\mathcal{F}}E\right) = \dot{\theta}\left[mr_0^2(a^2+1)e^{-2a\theta}(\ddot{\theta} - a\dot{\theta}^2) + mgr_0 e^{-a\theta}(\cos\theta - a\sin\theta)\right] = 0 \tag{3.195}$$

Now since $\dot{\theta} \neq 0$ as a function of time (otherwise the particle would not be moving), the term in the square brackets must be zero, i.e.,

$$mr_0^2(a^2+1)e^{-2a\theta}(\ddot{\theta} - a\dot{\theta}^2) + mgr_0 e^{-a\theta}(\cos\theta - a\sin\theta) = 0 \tag{3.196}$$

Then, dividing Eq. (3.196) by $mr_0^2(a^2+1)e^{-2a\theta}$, we obtain the differential equation of motion as

$$\ddot{\theta} - a\dot{\theta}^2 + \frac{g}{r_0(a^2+1)}e^{a\theta}(\cos\theta - a\sin\theta) = 0 \tag{3.197}$$

It is seen that the result of Eq. (3.197) is identical to that obtained in part (a).

Question 3–7

A particle of mass m slides without friction along the inner surface of a fixed cone of semi-vertex angle β as shown in the Fig. P3-7. The equation for the cone is given in cylindrical coordinates as

$$z = r \cot \beta$$

Knowing that the basis $\{\mathbf{E}_x, \mathbf{E}_y, \mathbf{E}_z\}$ is fixed to the cone, that θ is the angle between the $\mathbf{E}_x$-direction and the direction OQ where Q is the projection of the particle into the $\{\mathbf{E}_x, \mathbf{E}_y\}$-plane, and that gravity acts vertically downward, determine a system of two differential equations in terms of r and θ that describe the motion of the particle.

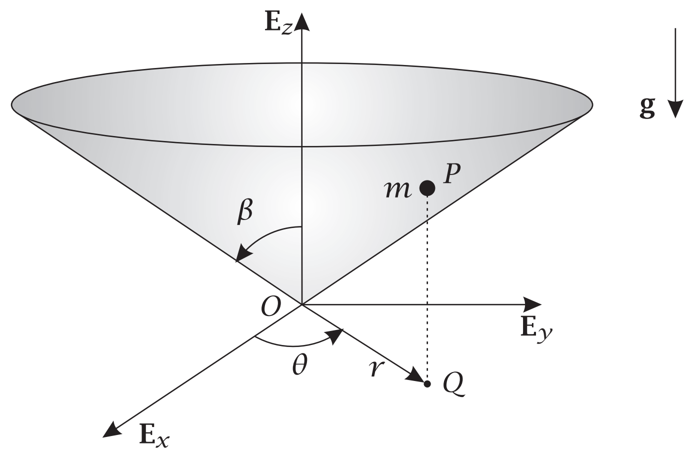


Figure P3-7

Solution to Question 3–7

Kinematics

First, let $\mathcal{F}$ be a reference frame fixed to the cone. Then, choose the following coordinate system fixed in reference frame $\mathcal{F}$:

	Origin at O	
$\mathbf{E}_x$	=	As Given
$\mathbf{E}_y$	=	As Given
$\mathbf{E}_z$	=	$\mathbf{E}_x \times \mathbf{E}_y$ = As Given

Next, let $\mathcal{A}$ be a reference frame fixed to the plane formed by the points O, Q, and P. Then, choose the following coordinate system fixed in reference frame

$\mathcal{A}$:

$$\begin{array}{ccc} & \text{Origin at } O & \\ \mathbf{e}_r & = & \text{Along } OQ \\ \mathbf{e}_z & = & \mathbf{E}_z \\ \mathbf{e}_\theta & = & \mathbf{E}_z \times \mathbf{e}_r \end{array}$$

The position of the particle is then given as

$$\mathbf{r} = r\mathbf{e}_r + z\mathbf{e}_z = r\mathbf{e}_r + r\cot\beta\mathbf{e}_z \tag{3.198}$$

Furthermore, the angular velocity of reference frame $\mathcal{A}$ in reference frame $\mathcal{F}$ is given as

$$^{\mathcal{F}}\boldsymbol{\omega}^{\mathcal{A}} = \dot{\theta}\mathbf{E}_z \tag{3.199}$$

The velocity of the particle in reference frame $\mathcal{F}$ is then obtained from the rate of change transport theorem as

$$^{\mathcal{F}}\mathbf{v} = \frac{^{\mathcal{F}}d\mathbf{r}}{dt} = \frac{^{\mathcal{A}}d\mathbf{r}}{dt} + {}^{\mathcal{F}}\boldsymbol{\omega}^{\mathcal{A}} \times \mathbf{r} \tag{3.200}$$

Now we have that

$$\frac{^{\mathcal{A}}d\mathbf{r}}{dt} = \dot{r}\mathbf{e}_r + \dot{r}\cot\beta\mathbf{e}_z \tag{3.201}$$

$$^{\mathcal{F}}\boldsymbol{\omega}^{\mathcal{A}} \times \mathbf{r} = \dot{\theta}\mathbf{e}_z \times (r\mathbf{e}_r + r\cot\beta\mathbf{e}_z) = r\dot{\theta}\mathbf{e}_\theta \tag{3.202}$$

Adding the expressions in Eq. (3.201) and Eq. (3.202), we obtain the velocity of the particle in reference frame $\mathcal{F}$ as

$$^{\mathcal{F}}\mathbf{v} = \dot{r}\mathbf{e}_r + r\dot{\theta}\mathbf{e}_\theta + \dot{r}\cot\beta\mathbf{e}_z \tag{3.203}$$

Next, applying the rate of change transport theorem to $^{\mathcal{F}}\mathbf{v}$, we obtain the acceleration of the particle in reference frame $\mathcal{F}$ as

$$^{\mathcal{F}}\mathbf{a} = \frac{^{\mathcal{F}}d}{dt}\left(^{\mathcal{F}}\mathbf{v}\right) = \frac{^{\mathcal{A}}d}{dt}\left(^{\mathcal{F}}\mathbf{v}\right) + {}^{\mathcal{F}}\boldsymbol{\omega}^{\mathcal{A}} \times {}^{\mathcal{F}}\mathbf{v} \tag{3.204}$$

Now we have that

$$\frac{^{\mathcal{A}}d}{dt}\left(^{\mathcal{F}}\mathbf{v}\right) = \ddot{r}\mathbf{e}_r + (\dot{r}\dot{\theta} + r\ddot{\theta})\mathbf{e}_\theta + \ddot{r}\cot\beta\mathbf{e}_z \tag{3.205}$$

$$^{\mathcal{F}}\boldsymbol{\omega}^{\mathcal{A}} \times {}^{\mathcal{F}}\mathbf{v} = \dot{\theta}\mathbf{E}_z \times \left(\dot{r}\mathbf{e}_r + r\dot{\theta}\mathbf{e}_\theta + \dot{r}\cot\beta\mathbf{E}_z\right) = \dot{r}\dot{\theta}\mathbf{e}_\theta - r\dot{\theta}^2\mathbf{e}_r \tag{3.206}$$

Adding the expressions in Eq. (3.205) and Eq. (3.206), we obtain the acceleration of the particle in reference frame $\mathcal{F}$ as

$$^{\mathcal{F}}\mathbf{a} = (\ddot{r} - r\dot{\theta}^2)\mathbf{e}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\mathbf{e}_\theta + \ddot{r}\cot\beta\mathbf{e}_z \tag{3.207}$$

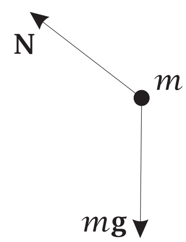


Figure 3-6 Free Body Diagram of Particle for Question 3–7.

Kinetics

In order to determine the differential equations of motion, we need to apply Newton's 2^{nd} Law, i.e., we need to apply $\mathbf{F} = m^{\mathcal{F}}\mathbf{a}$. The free body diagram of the particle is shown in Fig. 3-6. Using Fig. 3-6, we see that the only two forces acting on the particle are

$$
\begin{array}{rcl}
\mathbf{N} & = & \text{Reaction Force of Cone on Particle} \\
m\mathbf{g} & = & \text{Force of Gravity}
\end{array}
$$

Since we now that $\mathbf{N}$ must lie in the direction orthogonal to the surface of the cone while the force of gravity acts vertically downward, we can write $\mathbf{N}$ and $m\mathbf{g}$, respectively, as

$$
\mathbf{N} = N\mathbf{n} \tag{3.208}
$$

$$
m\mathbf{g} = -mg\mathbf{e}_z \tag{3.209}
$$

where $\mathbf{n}$ is the direction orthogonal to the surface of the cone at the location of the particle. Now we know that the direction orthogonal to the surface of the cone is the same as the direction of the *gradient* of the function that describes the cone. In particular, the function that describes the surface of the cone is given as

$$
z = r\cot\beta \tag{3.210}
$$

Rearranging Eq. (3.210), the function that describes the surface of the cone is given in cylindrical coordinates as

$$
f(r,\theta,z) = z - r\cot\beta = 0 \tag{3.211}
$$

The gradient of f in cylindrical coordinates is then given as

$$
\nabla f = \frac{\partial f}{\partial r}\mathbf{e}_r + \frac{1}{r}\frac{\partial f}{\partial \theta}\mathbf{e}_\theta + \frac{\partial f}{\partial z}\mathbf{e}_z \tag{3.212}
$$

where

$$\frac{\partial f}{\partial r} = -\cot\beta \quad (3.213)$$
$$\frac{\partial f}{\partial \theta} = 0 \quad (3.214)$$
$$\frac{\partial f}{\partial z} = 1 \quad (3.215)$$

We then obtain ∇f as

$$\nabla f = -\cot\beta\mathbf{e}_r + \mathbf{e}_z \quad (3.216)$$

The unit normal to the surface of the cone is then given as

$$\mathbf{n} = \frac{\nabla f}{\|\nabla f\|} = \frac{-\cot\beta\mathbf{e}_r + \mathbf{e}_z}{\sqrt{1+\cot^2\beta}} \quad (3.217)$$

Now from trigonometry we have that

$$1 + \cot^2\beta = \csc^2\beta = 1/\sin^2\beta \quad (3.218)$$

Substituting the result of Eq. (3.218) into Eq. (3.217), we obtain the unit normal to the surface of the cone as

$$\mathbf{n} = \sin\beta(-\cot\beta\mathbf{e}_r + \mathbf{e}_z) = -\cos\beta\mathbf{e}_r + \sin\beta\mathbf{e}_z \quad (3.219)$$

Then, substituting the expression for $\mathbf{n}$ from Eq. (3.219) into Eq. (3.208), we obtain the reaction force of the cone on the particle as

$$\mathbf{N} = N(-\cos\beta\mathbf{e}_r + \sin\beta\mathbf{e}_z) = -N\cos\beta\mathbf{e}_r + N\sin\beta\mathbf{e}_z \quad (3.220)$$

The resultant force on the particle is then given as

$$\mathbf{F} = \mathbf{N} + m\mathbf{g} = -N\cos\beta\mathbf{e}_r + N\sin\beta\mathbf{e}_z - mg\mathbf{e}_z = -N\cos\beta\mathbf{e}_r + (N\sin\beta - mg)\mathbf{e}_z \quad (3.221)$$

Setting $\mathbf{F}$ in Eq. (3.221) equal to $m{}^{\mathcal{F}}\mathbf{a}$ using the expression for ${}^{\mathcal{F}}\mathbf{a}$ from Eq. (3.207), we have that

$$-N\cos\beta\mathbf{e}_r + (N\sin\beta - mg)\mathbf{e}_z = m(\ddot{r} - r\dot{\theta}^2)\mathbf{e}_r + m(r\ddot{\theta} + 2\dot{r}\dot{\theta})\mathbf{e}_\theta + m\ddot{r}\cot\beta\mathbf{e}_z \quad (3.222)$$

Equating components in Eq. (3.222), we obtain the following three scalar equations:

$$-N\cos\beta = m(\ddot{r} - r\dot{\theta}^2) \quad (3.223)$$
$$0 = m(r\ddot{\theta} + 2\dot{r}\dot{\theta}) \quad (3.224)$$
$$N\sin\beta - mg = m\ddot{r}\cot\beta \quad (3.225)$$

Now, it is seen that Eq. (3.224) has no unknown reaction forces. Consequently, Eq. (3.224) is one of the differential equations of motion. Dropping m from Eq. (3.224), the first differential equation of motion can be written as

$$r\ddot{\theta} + 2\dot{r}\dot{\theta} = 0 \tag{3.226}$$

The second differential equation of motion can be obtained using Eq. (3.223) and Eq. (3.223). In particular, we can rearrange Eq. (3.225) as

$$N\sin\beta = m\ddot{r}\cot\beta + mg \tag{3.227}$$

Then, dividing Eq. (3.227) by Eq. (3.223), we obtain

$$\frac{N\sin\beta}{-N\cos\beta} = \frac{m\ddot{r}\cot\beta + mg}{m(\ddot{r} - r\dot{\theta}^2)} \tag{3.228}$$

Eq. (3.228) simplifies to

$$\tan\beta = -\frac{\ddot{r}\cot\beta + g}{\ddot{r} - r\dot{\theta}^2} \tag{3.229}$$

Rearranging Eq. (3.229) gives

$$(\ddot{r} - r\dot{\theta}^2)\tan\beta = -\ddot{r}\cot\beta - g \tag{3.230}$$

Then, dividing Eq. (3.230) by $\tan\beta$, we obtain

$$(\ddot{r} - r\dot{\theta}^2) = \ddot{r}\cot^2\beta + g\cot\beta = 0 \tag{3.231}$$

Rearranging Eq. (3.231) gives

$$(1 + \cot^2\beta)\ddot{r} - r\dot{\theta}^2 + g\cot\beta = 0 \tag{3.232}$$

Once again, using the fact that $1 + \cot^2\beta = \csc^2\beta$, Eq. (3.232) simplifies to

$$\csc^2\beta\ddot{r} - r\dot{\theta}^2 + g\cot\beta = 0 \tag{3.233}$$

Dividing Eq. (3.233) by $\csc^2\beta$, we obtain the second differential equation of motion as

$$\ddot{r} - r\dot{\theta}^2\sin^2\beta + g\cos\beta\sin\beta = 0 \tag{3.234}$$

The two differential equations that govern the motion of the particle are then given from Eq. (3.226) and Eq. (3.234) as

$$r\ddot{\theta} + 2\dot{r}\dot{\theta} = 0 \tag{3.235}$$

$$\ddot{r} - r\dot{\theta}^2\sin^2\beta + g\cos\beta\sin\beta = 0 \tag{3.236}$$

Question 3–9

A particle of mass m is attached to one end of a flexible but inextensible massless rope as shown in Fig. P3-9. The rope is wrapped around a cylinder of radius R where the cylinder rotates with constant angular velocity $\boldsymbol{\Omega}$ relative to the ground. The rope unravels from the cylinder in such a manner that it never becomes slack. Furthermore, point A is *fixed to the cylinder* and corresponds to a configuration where no portion of the rope is exposed while point B is the *instantaneous* point of contact of the exposed portion of the rope with the cylinder. Knowing that the exposed portion of the rope is tangent to the cylinder at every instant of the motion, that θ is the angle between points A and B, and assuming the initial conditions $\theta(t=0)=0$, $\dot{\theta}(t=0)=\Omega$ (where $\Omega=\|\boldsymbol{\Omega}\|$), determine (a) the angular velocity of the exposed portion of the rope as viewed by an observer fixed to the ground, (b) the acceleration of the particle as viewed by an observer fixed to the ground, (c) the differential equation for the particle in terms of the variable θ, and (d) the tension in the rope as a function of time.

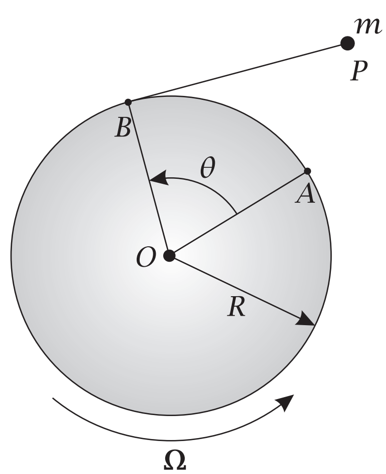


Figure P3-9

Solution to Question 3–9

Kinematics

First, let $\mathcal{F}$ be a reference frame that is fixed in the ground. Then, choose the following coordinate system fixed in reference frame $\mathcal{F}$:

	Origin at O	
$\mathbf{E}_x$	$=$	Along OA at $t=0$
$\mathbf{E}_z$	$=$	Out of Page
$\mathbf{E}_y$	$=$	$\mathbf{E}_z \times \mathbf{E}_x$

Next, let $\mathcal{A}$ be a reference frame fixed to the cylinder. Then, choose the following coordinate system fixed in reference frame $\mathcal{A}$:

	Origin at O	
$\mathbf{u}_x$	=	Along OA
$\mathbf{u}_z$	=	Out of Page
$\mathbf{u}_y$	=	$\mathbf{u}_z \times \mathbf{u}_x$

Finally, let $\mathcal{B}$ be a reference frame fixed to the rope. Then, choose the following coordinate system fixed in reference frame $\mathcal{B}$:

	Origin at B	
$\mathbf{e}_x$	=	Along OB
$\mathbf{e}_z$	=	Out of Page
$\mathbf{e}_y$	=	$\mathbf{e}_z \times \mathbf{e}_x$

Now, we note that the cylinder rotates with constant angular velocity $\boldsymbol{\Omega}$ about the $\mathbf{u}_z$-direction. Consequently, the angular velocity of the cylinder in reference frame $\mathcal{F}$ is given as

$$^{\mathcal{F}}\boldsymbol{\omega}^{\mathcal{A}} = \boldsymbol{\Omega} = \Omega \mathbf{u}_z \tag{3.237}$$

Next, since $\mathcal{A}$ is fixed in the cylinder and $\mathcal{B}$ is fixed in the rope, the angular velocity of the rope relative to the cylinder is equivalent to the angular velocity of reference frame $\mathcal{B}$ relative to reference frame $\mathcal{A}$. Observing from Fig. 3-7 that θ defines the rotation of the rope relative to the cylinder, we have that

$$^{\mathcal{A}}\boldsymbol{\omega}^{\mathcal{B}} = \dot{\theta}\mathbf{e}_z \tag{3.238}$$

Then, applying the angular velocity addition theorem, the angular velocity of the rope relative to the ground is obtained by adding the results of Eq. (3.237) and Eq. (3.238) to obtain

$$^{\mathcal{F}}\boldsymbol{\omega}^{\mathcal{B}} = {}^{\mathcal{F}}\boldsymbol{\omega}^{\mathcal{A}} + {}^{\mathcal{A}}\boldsymbol{\omega}^{\mathcal{B}} = \Omega\mathbf{u}_z + \dot{\theta}\mathbf{e}_z = (\Omega + \dot{\theta})\mathbf{e}_z \tag{3.239}$$

where we note that $\mathbf{u}_z = \mathbf{e}_z$. Next, we know that, when no portion of the rope is exposed (i.e., $s = 0$), the particle is in contact with point A on the cylinder. Using Fig. 3-7 along with the fact that the cylinder is circular, the arclength along the cylinder from point A to point B is given as

$$s = R\theta \tag{3.240}$$

Differentiating Eq. (3.240), we obtain

$$\dot{s} = R\dot{\theta} \tag{3.241}$$

Next, the position of the particle is given in terms of the basis $\{\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z\}$ as

$$\mathbf{r} = R\mathbf{e}_x - s\mathbf{e}_y = R\mathbf{e}_x - R\theta\mathbf{e}_y \tag{3.242}$$

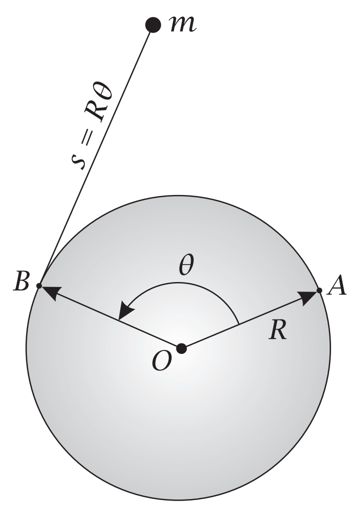


Figure 3-7 Geometry of Rope and Cylinder for Question 3–9.

The velocity of the particle in reference frame $\mathcal{F}$ is then given as

$$ {}^{\mathcal{F}}\mathbf{v} = \frac{{}^{\mathcal{F}}d\mathbf{r}}{dt} = \frac{{}^{\mathcal{B}}d\mathbf{r}}{dt} + {}^{\mathcal{F}}\boldsymbol{\omega}^{\mathcal{B}} \times \mathbf{r} \tag{3.243} $$

where

$$ \begin{aligned} \frac{{}^{\mathcal{B}}d\mathbf{r}}{dt} &= -R\dot{\theta}\mathbf{e}_y && (3.244) \\ {}^{\mathcal{F}}\boldsymbol{\omega}^{\mathcal{B}} \times \mathbf{r} &= \left(\Omega + \dot{\theta}\right)\mathbf{e}_z \times \left(R\mathbf{e}_x - R\theta\mathbf{e}_y\right) \\ &= R\theta\left(\Omega + \dot{\theta}\right)\mathbf{e}_x + R\left(\Omega + \dot{\theta}\right)\mathbf{e}_y && (3.245) \end{aligned} $$

Adding the expressions in Eq. (3.244) and Eq. (3.245), we obtain the velocity of the particle in reference frame $\mathcal{F}$ as

$$ {}^{\mathcal{F}}\mathbf{v} = -R\dot{\theta}\mathbf{e}_y + R\theta\left(\Omega + \dot{\theta}\right)\mathbf{e}_x + R\left(\Omega + \dot{\theta}\right)\mathbf{e}_y \tag{3.246} $$

Simplifying Eq. (3.246), we obtain

$$ {}^{\mathcal{F}}\mathbf{v} = R\theta\left(\Omega + \dot{\theta}\right)\mathbf{e}_x + R\Omega\mathbf{e}_y \tag{3.247} $$

Then, the acceleration in reference frame $\mathcal{F}$ is given as

$$ {}^{\mathcal{F}}\mathbf{v} = \frac{{}^{\mathcal{F}}d}{dt}\left({}^{\mathcal{F}}\mathbf{v}\right) = \frac{{}^{\mathcal{B}}d}{dt}\left({}^{\mathcal{F}}\mathbf{v}\right) + {}^{\mathcal{F}}\boldsymbol{\omega}^{\mathcal{B}} \times {}^{\mathcal{F}}\mathbf{v} \tag{3.248} $$

where

$$ \begin{aligned} \frac{{}^{\mathcal{B}}d}{dt}\left({}^{\mathcal{F}}\mathbf{v}\right) &= \left[R\dot{\theta}\left(\Omega + \dot{\theta}\right) + R\theta\ddot{\theta}\right]\mathbf{e}_x && (3.249) \\ {}^{\mathcal{F}}\boldsymbol{\omega}^{\mathcal{B}} \times {}^{\mathcal{F}}\mathbf{v} &= \left(\Omega + \dot{\theta}\right)\mathbf{e}_z \times \left[R\theta\left(\Omega + \dot{\theta}\right)\mathbf{e}_x + R\Omega\mathbf{e}_y\right] \end{aligned} $$

$$= \quad -R\Omega\left(\Omega+\dot{\theta}\right)\mathbf{e}_x + R\theta\left(\Omega+\dot{\theta}\right)^2\mathbf{e}_y \tag{3.250}$$

Adding Eq. (3.249) and Eq. (3.250), we obtain the acceleration of the particle in reference frame $\mathcal{F}$

$$^{\mathcal{F}}\mathbf{a} = \left[R\dot{\theta}\left(\Omega+\dot{\theta}\right)+R\theta\ddot{\theta}\right]\mathbf{e}_x - R\Omega\left(\Omega+\dot{\theta}\right)\mathbf{e}_x + R\theta\left(\Omega+\dot{\theta}\right)^2\mathbf{e}_y \tag{3.251}$$

Simplifying Eq. (3.251

$$^{\mathcal{F}}\mathbf{a} = \left[R\dot{\theta}^2+R\theta\ddot{\theta}-R\Omega^2\right]\mathbf{e}_x + R\theta\left(\Omega+\dot{\theta}\right)^2\mathbf{e}_y \tag{3.252}$$

Kinetics and Differential Equation of Motion

We need to apply Newton's 2^{nd} law to the particle. Using the free body diagram as shown in Fig. 3-8, it can be seen that the only force acting on the particle is due to the tension in the rope. Since the tension must act along the direction of the rope, we have that

$$\mathbf{T} = T\mathbf{e}_y \tag{3.253}$$

Therefore, the resultant force acting on the particle is given as

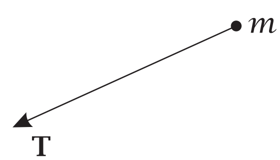


Figure 3-8 Free Body Diagram for Question 3.5

$$\mathbf{F} = \mathbf{T} = T\mathbf{e}_y \tag{3.254}$$

Setting $\mathbf{F} = m\,^{\mathcal{F}}\mathbf{a}$ using $^{\mathcal{F}}\mathbf{a}$ from Eq. (3.252), we obtain

$$T\mathbf{e}_y = m\left[R\dot{\theta}^2+R\theta\ddot{\theta}-R\Omega^2\right]\mathbf{e}_x + mR\theta\left(\Omega+\dot{\theta}\right)^2\mathbf{e}_y \tag{3.255}$$

We then obtain the following two scalar equations:

$$m\left[R\dot{\theta}^2+R\theta\ddot{\theta}-R\Omega^2\right] = 0 \tag{3.256}$$

$$mR\theta\left(\Omega+\dot{\theta}\right)^2 = T \tag{3.257}$$

From Eq (3.256) we have

$$R\dot{\theta}^2+R\theta\ddot{\theta}-R\Omega^2 = 0 \tag{3.258}$$

Simplifying this last expression, we obtain the differential equation of motion as

$$\dot{\theta}^2+\theta\ddot{\theta}-\Omega^2 = 0 \tag{3.259}$$

Tension in Rope As a Function of Time

Eq. (3.259) can be solved for θ. This is done as follows. First, we note that

$$\dot{\theta}^2 + \theta\ddot{\theta} = \frac{d}{dt}\left(\theta\dot{\theta}\right) \tag{3.260}$$

Substituting this last result into Eq. (3.259), we obtain

$$\frac{d}{dt}\left(\theta\dot{\theta}\right) - \Omega^2 = 0 \tag{3.261}$$

Integrating Eq. (3.261) once with respect to time, we obtain

$$\theta\dot{\theta} = \Omega^2 t + c_1 \tag{3.262}$$

where c_1 is an arbitrary constant of integration. Then, applying the initial condition $\theta(t = 0) = 0$, we have that

$$c_1 = 0 \tag{3.263}$$

Therefore, we have

$$\theta\dot{\theta} = \Omega^2 t \tag{3.264}$$

Then, separating variables in Eq. (3.264), we obtain

$$\theta d\theta = \Omega^2 t dt \tag{3.265}$$

Integrating both sides of Eq. (3.265) gives

$$\frac{\theta^2}{2} = \frac{\Omega^2 t^2}{2} + c_2 \tag{3.266}$$

where c_2 is an arbitrary constant of integration. Then, again applying the initial condition $\theta(t = 0) = 0$, we obtain

$$c_2 = 0 \tag{3.267}$$

Consequently,

$$\frac{\theta^2}{2} = \frac{\Omega^2 t^2}{2} \tag{3.268}$$

which gives

$$\theta^2 = \Omega^2 t^2 \tag{3.269}$$

Since θ has to be positive, we can take the principal square root of Eq. (3.269) to obtain

$$\theta = \Omega t \tag{3.270}$$

Differentiating with respect to time, we have

$$\dot{\theta} = \Omega \tag{3.271}$$

Then, substituting θ from Eq. (3.270) and $\dot{\theta}$ from Eq. (3.271) into Eq. 3.257 gives

$$T = mR\Omega t\,(\Omega + \Omega)^2 = 4mR\Omega^3 t \tag{3.272}$$

The tension in the rope is then given as

$$\mathbf{T} = \left[4mR\Omega^3 t\right]\mathbf{e}_y \tag{3.273}$$

Question 3–10

A particle of mass m moves under the influence of gravity in the vertical plane along a track as shown in Fig. P3-10. The equation for the track is given in Cartesian coordinates as

$$y = -\ln\cos x$$

where $-\pi/2 < x < \pi/2$. Using the horizontal component of position, x, as the variable to describe the motion determine the differential equation of motion for the particle using (a) Newton's 2^{nd} law and (b) one of the forms of the work-energy theorem for a particle.

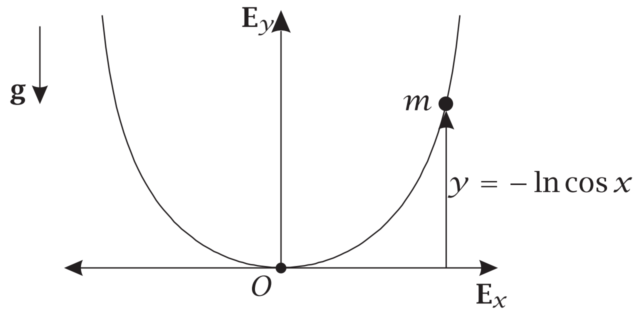


Figure P3-10

Solution to Question 3–10

Kinematics

For this problem, it is convenient to use a reference frame $\mathcal{F}$ that is fixed to the track. Then, we choose the following coordinate system fixed in reference frame $\mathcal{F}$:

	Origin at O	
$\mathbf{E}_x$	$=$	Along Ox
$\mathbf{E}_y$	$=$	Along Oy
$\mathbf{E}_z$	$=$	$\mathbf{E}_x \times \mathbf{E}_y$

The position of the particle is then given as

$$\mathbf{r} = x\mathbf{E}_x - \ln\cos x\,\mathbf{E}_y \tag{3.274}$$

Now, since the basis $\{\mathbf{E}_x, \mathbf{E}_y, \mathbf{E}_z\}$ does not rotate, the velocity in reference frame $\mathcal{F}$ is given as

$${}^{\mathcal{F}}\mathbf{v} = \dot{x}\mathbf{E}_x + \dot{x}\tan x\,\mathbf{E}_y \tag{3.275}$$

Using the velocity from Eq. (3.275), the speed of the particle in reference frame $\mathcal{F}$ is given as

$${}^{\mathcal{F}}v = \|{}^{\mathcal{F}}\mathbf{v}\| = \dot{x}\sqrt{1+\tan^2 x} = \dot{x}\sec x \tag{3.276}$$

Arclength Parameter as a Function of x

Now we recall the arclength equation as

$$\frac{d}{dt}\left({}^{\mathcal{F}}s\right) = {}^{\mathcal{F}}v = \dot{x}\sqrt{1+\tan^2 x} = \dot{x}\sec x \tag{3.277}$$

Separating variables in Eq. (3.277), we obtain

$$^{\mathcal{F}}ds = \sec x\,dx \tag{3.278}$$

Integrating both sides of Eq. (3.278) gives

$$^{\mathcal{F}}s - {}^{\mathcal{F}}s_0 = \int_{x_0}^{x} \sec x\,dx \tag{3.279}$$

Using the integral given for $\sec x$, we obtain

$$^{\mathcal{F}}s - {}^{\mathcal{F}}s_0 = \ln\left[\sec x + \tan x\right]_{x_0}^{x} = \ln\left[\frac{\sec x + \tan x}{\sec x_0 + \tan x_0}\right] \tag{3.280}$$

Noting that ${}^{\mathcal{F}}s(0) = {}^{\mathcal{F}}s_0 = 0$, the arclength is given as

$$^{\mathcal{F}}s = \ln\left[\sec x + \tan x\right]_{x_0}^{x} \tag{3.281}$$

Simplifying Eq. (3.281), we obtain

$$^{\mathcal{F}}s = \ln\left[\frac{\sec x + \tan x}{\sec x_0 + \tan x_0}\right] \tag{3.282}$$

Intrinsic Basis

Next, we need to compute the intrinsic basis. First, we have the tangent vector as

$$\mathbf{e}_t = \frac{^{\mathcal{F}}\mathbf{v}}{^{\mathcal{F}}v} = \frac{\dot{x}(\mathbf{E}_x + \tan x\mathbf{E}_y)}{\dot{x}\sec x} = \frac{1}{\sec x}\mathbf{E}_x + \frac{\tan x}{\sec x}\mathbf{E}_y \tag{3.283}$$

Now we note that $\sec x = 1/\cos x$. Therefore,

$$\frac{\tan x}{\sec x} = \sin x \tag{3.284}$$

Eq. (3.283) then simplifies to

$$\mathbf{e}_t = \cos x\mathbf{E}_x + \sin x\mathbf{E}_y \tag{3.285}$$

Next, the principle unit normal is given as

$$\frac{^{\mathcal{F}}d\mathbf{e}_t}{dt} = \kappa\,{}^{\mathcal{F}}v\,\mathbf{e}_n \tag{3.286}$$

Differentiating $\mathbf{e}_t$ in Eq. (3.285), we obtain

$$\frac{{}^{\mathcal{F}}d\mathbf{e}_t}{dt} = -\dot{x}\sin x\mathbf{E}_x + \dot{x}\cos x\mathbf{E}_y \tag{3.287}$$

Consequently,

$$\left\|\frac{{}^{\mathcal{F}}d\mathbf{e}_t}{dt}\right\| = \dot{x} = \kappa{}^{\mathcal{F}}v \tag{3.288}$$

which implies that

$$\mathbf{e}_n = \frac{{}^{\mathcal{F}}d\mathbf{e}_t/dt}{\left\|{}^{\mathcal{F}}d\mathbf{e}_t/dt\right\|} = \frac{-\dot{x}\sin x\mathbf{E}_x + \dot{x}\cos x\mathbf{E}_y}{\dot{x}} = -\sin x\mathbf{E}_x + \cos x\mathbf{E}_y \tag{3.289}$$

Then, using ${}^{\mathcal{F}}v$ from Eq. (3.276), we obtain the curvature as

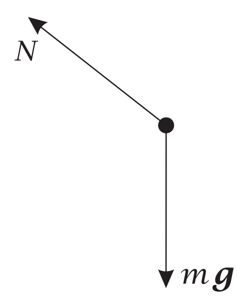
$$\kappa = \frac{\dot{x}}{\dot{x}\sec x} = \frac{1}{\sec x} = \cos x \tag{3.290}$$

Finally, the principle unit bi-normal is given as

$$\mathbf{e}_b = \mathbf{e}_t \times \mathbf{e}_n = (\cos x\mathbf{E}_x + \sin x\mathbf{E}_y) \times (-\sin x\mathbf{E}_x + \cos x\mathbf{E}_y) = \mathbf{E}_z \tag{3.291}$$

Differential Equation of Motion in Terms of x

The differential equation of motion is obtained using Newton's 2^{nd} Law for a particle. First we obtain $\mathbf{F}$ using the free body diagram shown below: Using the



free body diagram, we can see that

$$\begin{aligned}\mathbf{N} &= N\mathbf{e}_n \\ m\mathbf{g} &= -mg\mathbf{E}_y\end{aligned}$$

Therefore, the resultant force acting on the particle is

$$\mathbf{F} = \mathbf{N} + m\mathbf{g} = N\mathbf{e}_n - mg\mathbf{E}_y \tag{3.292}$$

Next, the acceleration is given in the intrinsic basis as

$${}^{\mathcal{F}}\mathbf{a} = \frac{d}{dt}\left({}^{\mathcal{F}}v\right)\mathbf{e}_t + \kappa\left({}^{\mathcal{F}}v\right)\mathbf{e}_n \tag{3.293}$$

Now, using ${}^{\mathcal{F}}v$ from Eq. (3.276), we obtain $d({}^{\mathcal{F}}v)/dt$ as

$$\frac{d}{dt}\left({}^{\mathcal{F}}v\right) = \ddot{x}\sec x + \dot{x}^2\sec x\tan x = \sec x\left[\ddot{x} + \dot{x}^2\tan x\right] \tag{3.294}$$

Then, using κ from Eq. (3.290) we obtain

$$\kappa\left({}^{\mathcal{F}}v\right) = \cos x(\dot{x}\sec x)^2 = \dot{x}^2\sec x \tag{3.295}$$

The acceleration of the particle in reference frame $\mathcal{F}$ is then given as

$${}^{\mathcal{F}}\mathbf{a} = \sec x\left[\ddot{x} + \dot{x}^2\tan x\right]\mathbf{e}_t + \dot{x}^2\sec x\mathbf{e}_n \tag{3.296}$$

Setting $\mathbf{F}$ from Eq. (3.292) equal to $m{}^{\mathcal{F}}\mathbf{a}$ using ${}^{\mathcal{F}}\mathbf{a}$ from Eq. (3.296), we obtain

$$N\mathbf{e}_n - mg\mathbf{E}_y = m\sec x\left[\ddot{x} + \dot{x}^2\tan x\right]\mathbf{e}_t + m\dot{x}^2\sec x\mathbf{e}_n \tag{3.297}$$

Now we can take the scalar products on both sides of Eq. (3.297) in the $\mathbf{e}_t$ and en directions. Taking the scalar product on both sides of Eq. (3.297) in the $\mathbf{e}_t$-direction, we obtain

$$-mg\mathbf{E}_y\cdot\mathbf{e}_n = m\sec x\left[\ddot{x} + \dot{x}^2\tan x\right] \tag{3.298}$$

Taking the scalar product on both sides in the $\mathbf{e}_n$-direction, we obtain

$$N - mg\mathbf{E}_y\cdot\mathbf{e}_n = m\dot{x}^2\sec x \tag{3.299}$$

Now we note that

$$\begin{array}{rcl} \mathbf{E}_y\cdot\mathbf{e}_t & = & \mathbf{E}_y\cdot(\cos x\mathbf{E}_x + \sin x\mathbf{E}_y) = \sin x \\ \mathbf{E}_y\cdot\mathbf{e}_n & = & \mathbf{E}_y\cdot(-\sin x\mathbf{E}_x + \cos x\mathbf{E}_y) = \cos x \end{array} \tag{3.300}$$

Substituting $\mathbf{E}_y\cdot\mathbf{e}_t$ and $\mathbf{E}_y\cdot\mathbf{e}_n$ into Eq. (3.298) and Eq. (3.299), respectively, we obtain the following two scalar equations:

$$\begin{array}{rcl} m\sec x\left[\ddot{x} + \dot{x}^2\tan x\right] & = & -mg\sin x \\ m\dot{x}^2\sec x & = & N - mg\cos x \end{array} \tag{3.301}$$

Seeing that the first equation in Eq. (3.301) has no reaction forces, the differential equation of motion of the particle is given as

$$m\sec x\left[\ddot{x} + \dot{x}^2\tan x\right] = -mg\sin x \tag{3.302}$$

Eq. (3.302) can be rearranged to give

$$\ddot{x}\sec x + \dot{x}^2\sec x\tan x + g\sin x = 0 \tag{3.303}$$

Question 3–11

A particle of mass m moves in the horizontal plane as shown in Fig. P3-11. The particle is attached to a linear spring with spring constant K and unstretched length ℓ while the spring is attached at its other end to the fixed point O. Assuming no gravity, (a) determine a system of two differential equations of motion for the particle in terms of the variables r and θ, (b) show that the total energy of the system is conserved, and (c) show that the angular momentum relative to point O is conserved.

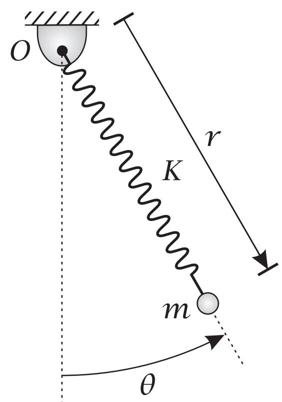


Figure P3-11

Solution to Question 3–11

Acceleration of Particle

First, let $\mathcal{F}$ be a reference frame fixed to the ground. Next, let $\mathcal{A}$ be a reference frame that is fixed to the direction Om. Corresponding to reference frame $\mathcal{A}$, we choose the following coordinate system to describe the motion of the particle:

	Origin at O (Corner)	
$\mathbf{e}_r$	=	Along Om
$\mathbf{E}_z$	=	Out of Page
$\mathbf{e}_\theta$	=	$\mathbf{E}_z \times \mathbf{e}_r$

The position of the particle is then given as

$$\mathbf{r} = r\mathbf{e}_r \tag{3.304}$$

Now the angular velocity of reference frame $\mathcal{A}$ in reference frame $\mathcal{F}$ is given as

$$^{\mathcal{F}}\boldsymbol{\omega}^{\mathcal{A}} = \dot{\theta}\mathbf{E}_z \tag{3.305}$$

The velocity in reference frame $\mathcal{F}$ is computed from the rate of change transport theorem as

$$^{\mathcal{F}}\mathbf{v} = \frac{^{\mathcal{F}}d\mathbf{r}}{dt} = \frac{^{\mathcal{A}}d\mathbf{r}}{dt} + {^{\mathcal{F}}\boldsymbol{\omega}^{\mathcal{A}}} \times \mathbf{r} \tag{3.306}$$

where

$$\begin{array}{rcl} \frac{^{\mathcal{A}}d\mathbf{r}}{dt} & = & \dot{r}\mathbf{e}_r \\ ^{\mathcal{F}}\boldsymbol{\omega}^{\mathcal{A}} \times \mathbf{r} & = & \dot{\theta}\mathbf{E}_z \times r\mathbf{e}_r = r\dot{\theta}\mathbf{e}_\theta \end{array} \tag{3.307}$$

Adding the two expressions in Eq. (3.307), we obtain $^{\mathcal{F}}\mathbf{v}$ as

$$^{\mathcal{F}}\mathbf{v} = \dot{r}\mathbf{e}_r + r\dot{\theta}\mathbf{e}_\theta \tag{3.308}$$

Applying the rate of change transport theorem to $^{\mathcal{F}}\mathbf{v}$, the acceleration of the particle in reference frame $\mathcal{F}$ is given as

$$^{\mathcal{F}}\mathbf{a} = \frac{^{\mathcal{F}}d}{dt}\left(^{\mathcal{F}}\mathbf{v}\right) = \frac{^{\mathcal{A}}d}{dt}\left(^{\mathcal{F}}\mathbf{v}\right) + {^{\mathcal{F}}\boldsymbol{\omega}^{\mathcal{A}}} \times {^{\mathcal{F}}\mathbf{v}} \tag{3.309}$$

where

$$\begin{array}{rcl} \frac{^{\mathcal{A}}d}{dt}\left(^{\mathcal{F}}\mathbf{v}\right) & = & \ddot{r}\mathbf{e}_r + (\dot{r}\dot{\theta} + r\ddot{\theta})\mathbf{e}_\theta \\ ^{\mathcal{F}}\boldsymbol{\omega}^{\mathcal{A}} \times {^{\mathcal{F}}\mathbf{v}} & = & \dot{\theta}\mathbf{E}_z \times (\dot{r}\mathbf{e}_r + r\dot{\theta}\mathbf{e}_\theta) = -r\dot{\theta}^2\mathbf{e}_r + \dot{r}\dot{\theta}\mathbf{e}_\theta \end{array} \tag{3.310}$$

Adding the two expressions in Eq. (3.310), we obtain $^{\mathcal{F}}\mathbf{a}$ as

$$^{\mathcal{F}}\mathbf{a} = (\ddot{r} - r\dot{\theta}^2)\mathbf{e}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\mathbf{e}_\theta \tag{3.311}$$

System of Two Differential Equations

To obtain a system of two differential equations, we need to apply Newton's 2^{nd} Law to the particle. We already have $^{\mathcal{F}}\mathbf{a}$ from Eq. (3.311). Next, in order to obtain an expression for the resultant force, $\mathbf{F}$, we need to examine the free body diagram as shown in Fig. 3-9 where $\mathbf{F}_s$ is the force due to the spring. Now

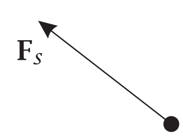


Figure 3-9 Free Body Diagram for Question 2.10.

we note that the general form for the force of a linear spring is

$$\mathbf{F}_s = -K(\ell - \ell_0)\mathbf{u}_s \tag{3.312}$$

Now since the attachment point of the spring for this problem is $\mathbf{r}_A = \mathbf{0}$, we have that

$$\ell = \|\mathbf{r} - \mathbf{r}_A\| = \|\mathbf{r}\| = r \tag{3.313}$$

Furthermore, the direction $\mathbf{u}_s$ is given as

$$\mathbf{u}_s = \frac{\mathbf{r} - \mathbf{r}_A}{\|\mathbf{r} - \mathbf{r}_A\|} = \frac{r\mathbf{e}_r}{r} = \mathbf{e}_r \tag{3.314}$$

Finally, the unstretched length of the spring is given as

$$\ell_0 = L \tag{3.315}$$

Therefore, we obtain the spring force as

$$\mathbf{F}_s = -K\,(r - L)\,\mathbf{e}_r \tag{3.316}$$

Next, since the only force acting on the particle is that of the spring, we can set $\mathbf{F}_s$ from Eq. (3.316) equal to $m^{\mathcal{F}}\mathbf{a}$ from Eq. (3.311) to give

$$-K\,[r - L]\,\mathbf{e}_r = m(\ddot{r} - r\dot{\theta}^2)\mathbf{e}_r + m(r\ddot{\theta} + 2\dot{r}\dot{\theta})\mathbf{e}_\theta \tag{3.317}$$

Equating components, we obtain the following two scalar equations:

$$\begin{array}{rcr} m(\ddot{r} - r\dot{\theta}^2) & = & -K\,[r - L] \\ m(r\ddot{\theta} + 2\dot{r}\dot{\theta}) & = & 0 \end{array} \tag{3.318}$$

Since there are no reaction forces in either of the equations in Eq. (3.318), these two equations *are* the differential equations of motion for the particle.

Conservation of Energy

From the work-energy theorem for a particle, we have that

$$\frac{d}{dt}\left({}^{\mathcal{F}}E\right) = \mathbf{F}^{nc} \cdot {}^{\mathcal{F}}\mathbf{v} \tag{3.319}$$

For this problem, the only force acting on the particle is that of the linear spring. Since the spring force is conservative, we have that $\mathbf{F}^{nc} = \mathbf{0}$. Therefore,

$$\frac{d}{dt}\left({}^{\mathcal{F}}E\right) = 0 \tag{3.320}$$

which implies that

$${}^{\mathcal{F}}E = \text{constant} \tag{3.321}$$

which implies that energy is conserved.