Question 3–12

A particle of mass m is attached to a linear spring with spring constant K and unstretched length r_0 as shown in Fig. P3-12. The spring is attached at its other end at point P to the free end of a rigid massless arm of length l. The arm is hinged at its other end and rotates in a circular path at a constant angular rate ω. Knowing that the angle θ is measured from the downward direction and assuming no friction, determine a system of two differential equations of motion for the particle in terms of r and θ.

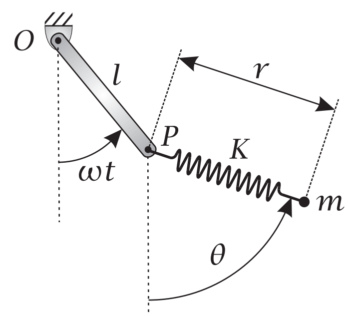

Figure P3-12

Solution to Question 3–12

Kinematics

First, let $\mathcal{F}$ be a fixed reference frame. Then, choose the following coordinate system fixed in reference frame $\mathcal{F}$:

	Origin at O	
$\mathbf{E}_x$	=	Along OP When $t = 0$
$\mathbf{E}_z$	=	Out of Page
$\mathbf{E}_y$	=	$\mathbf{E}_z \times \mathbf{E}_x$

Next, let $\mathcal{A}$ be a reference frame fixed to the arm. Then, choose the following coordinate system fixed in reference frame $\mathcal{A}$:

	Origin at O	
$\mathbf{e}_x$	=	Along OP
$\mathbf{e}_z$	=	Out of Page ($= \mathbf{E}_z$)
$\mathbf{e}_y$	=	$\mathbf{e}_z \times \mathbf{e}_x$

Finally, let $\mathcal{B}$ be a reference frame fixed to the direction along which the spring lies (i.e., the direction Pm). Then, choose the following coordinate system fixed in reference frame $\mathcal{B}$:

	Origin at O	
$\mathbf{u}_r$	=	Along Pm
$\mathbf{u}_z$	=	Out of Page ($= \mathbf{E}_z = \mathbf{e}_z$)
$\mathbf{u}_\theta$	=	$\mathbf{u}_z \times \mathbf{u}_r$

The geometry of the bases $\{\mathbf{E}_x, \mathbf{E}_y, \mathbf{E}_z\}$, $\{\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z\}$, and $\{\mathbf{u}_r, \mathbf{u}_\theta, \mathbf{u}_z\}$ is shown in Fig. 3-10. Using Fig. 3-10, we have the following relationship between the basis

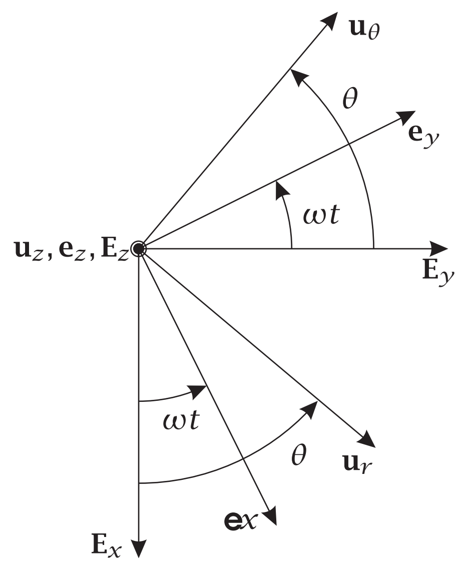

Figure 3-10 Geometry of Bases $\{\mathbf{E}_x, \mathbf{E}_y, \mathbf{E}_z\}$, $\{\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z\}$, and $\{\mathbf{u}_r, \mathbf{u}_\theta, \mathbf{u}_z\}$ for Question 3–12 .

$\{\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z\}$ and the basis $\{\mathbf{u}_r, \mathbf{u}_\theta, \mathbf{u}_z\}$:

$$
\begin{array}{rcl}
\mathbf{e}_x & = & \cos(\theta - \omega t)\mathbf{u}_r - \sin(\theta - \omega t)\mathbf{u}_\theta \\
\mathbf{e}_y & = & \sin(\theta - \omega t)\mathbf{u}_r + \cos(\theta - \omega t)\mathbf{u}_\theta
\end{array}
\tag{3.322}
$$

Next, observing that the basis $\{\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z\}$ rotates with angular rate ω relative to the basis $\{\mathbf{E}_x, \mathbf{E}_y, \mathbf{E}_z\}$, the angular velocity of reference frame $\mathcal{A}$ in reference frame $\mathcal{F}$ is given as

$$
{}^{\mathcal{F}}\boldsymbol{\omega}^{\mathcal{A}} = \omega \mathbf{E}_z = \omega \mathbf{e}_z \tag{3.323}
$$

Next, using Eq. (3.322), we observe that the angle formed between the basis vectors $\mathbf{u}_r$ and $\mathbf{e}_x$ (and similarly between $\mathbf{u}_\theta$ and $\mathbf{e}_y$) is $\theta - \omega t$. Consequently, the angular velocity of reference frame $\mathcal{B}$ in reference frame $\mathcal{A}$ is given as

$$
{}^{\mathcal{A}}\boldsymbol{\omega}^{\mathcal{B}} = (\dot{\theta} - \omega)\mathbf{e}_z = (\dot{\theta} - \omega)\mathbf{u}_z \tag{3.324}
$$

Finally, observing that the basis $\{\mathbf{u}_r, \mathbf{u}_\theta, \mathbf{u}_z\}$ rotates with angular rate $\dot{\theta}$ relative to the basis $\{\mathbf{E}_x, \mathbf{E}_y, \mathbf{E}_z\}$, we obtain the angular velocity of reference frame $\mathcal{B}$ in reference frame $\mathcal{F}$ as

$$^{\mathcal{F}}\boldsymbol{\omega}^{\mathcal{B}} = \dot{\theta}\mathbf{u}_z \tag{3.325}$$

The position of the particle can be written as

$$\mathbf{r} = \mathbf{r}_P + \mathbf{r}_{m/P} \tag{3.326}$$

where $\mathbf{r}_P$ is the position of point P and $\mathbf{r}_{m/P}$ is the postion of the particle relative to point P. In terms of the bases defined above, we have that

$$\begin{aligned} \mathbf{r}_P &= R\mathbf{e}_x \\ \mathbf{r}_{m/P} &= r\mathbf{u}_r \end{aligned} \tag{3.327}$$

Substituting the expressions from Eq. (3.327) into Eq. (3.326), we obtain

$$\mathbf{r} = R\mathbf{e}_x + r\mathbf{u}_r \tag{3.328}$$

Differentiating the expression for the position as given in Eq. (3.328) in reference frame $\mathcal{F}$, we have that

$$^{\mathcal{F}}\mathbf{v} = \frac{^{\mathcal{F}}d\mathbf{r}}{dt} = \frac{^{\mathcal{F}}d}{dt}(\mathbf{r}_P) + \frac{^{\mathcal{F}}d}{dt}(\mathbf{r}_{m/P}) = {^{\mathcal{F}}\mathbf{v}_P} + {^{\mathcal{F}}\mathbf{v}_{m/P}} \tag{3.329}$$

Now since $\mathbf{r}_P$ is expressed in the basis $\{\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z\}$ and $\{\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z\}$ is fixed in reference frame $\mathcal{A}$, we can apply the rate of change transport theorem to $\mathbf{r}_P$ between reference frames $\mathcal{A}$ and $\mathcal{F}$ to give

$$^{\mathcal{F}}\mathbf{v}_P = \frac{^{\mathcal{F}}d}{dt}(\mathbf{r}_P) = \frac{^{\mathcal{A}}d}{dt}(\mathbf{r}_P) + {^{\mathcal{F}}\boldsymbol{\omega}^{\mathcal{A}}} \times \mathbf{r}_P \tag{3.330}$$

Now we have that

$$\begin{aligned} \frac{^{\mathcal{A}}d}{dt}(\mathbf{r}_P) &= 0 \\ {^{\mathcal{F}}\boldsymbol{\omega}^{\mathcal{A}}} \times \mathbf{r}_P &= \omega\mathbf{e}_z \times R\mathbf{e}_x = R\omega\mathbf{e}_y \end{aligned} \tag{3.331}$$

Adding the two expressions in Eq. (3.331), we obtain

$$^{\mathcal{F}}\mathbf{v}_P = R\omega\mathbf{e}_y \tag{3.332}$$

Next, since $\mathbf{r}_{m/P}$ is expressed in the basis $\{\mathbf{u}_r, \mathbf{u}_\theta, \mathbf{u}_z\}$ and $\{\mathbf{u}_r, \mathbf{u}_\theta, \mathbf{u}_z\}$ is fixed in reference frame $\mathcal{B}$, we can apply the rate of change transport theorem to $\mathbf{r}_{m/P}$ between reference frames $\mathcal{B}$ and $\mathcal{F}$ to give

$$^{\mathcal{F}}\mathbf{v}_{m/P} = \frac{^{\mathcal{F}}d}{dt}(\mathbf{r}_{m/P}) = \frac{^{\mathcal{B}}d}{dt}(\mathbf{r}_{m/P}) + {^{\mathcal{F}}\boldsymbol{\omega}^{\mathcal{B}}} \times \mathbf{r}_{m/P} \tag{3.333}$$

Now we have that

$$\begin{aligned} \frac{{}^{\mathcal{B}}d}{dt}(\mathbf{r}_{m/P}) &= \dot{r}\mathbf{u}_r \\ {}^{\mathcal{F}}\boldsymbol{\omega}^{\mathcal{B}} \times \mathbf{r}_{m/P} &= \dot{\theta}\mathbf{u}_z \times r\mathbf{u}_r = r\dot{\theta}\mathbf{u}_\theta \end{aligned} \tag{3.334}$$

Adding the two expressions in Eq. (3.334), we have that

$$^{\mathcal{F}}\mathbf{v}_{m/P} = \dot{r}\mathbf{u}_r + r\dot{\theta}\mathbf{u}_\theta \tag{3.335}$$

Then, adding Eq. (3.332) and Eq. (3.335), we obtain the velocity of the particle in reference frame $\mathcal{F}$ as

$$^{\mathcal{F}}\mathbf{v} = {}^{\mathcal{F}}\mathbf{v}_P + {}^{\mathcal{F}}\mathbf{v}_{m/P} = R\omega\mathbf{e}_y + \dot{r}\mathbf{u}_r + r\dot{\theta}\mathbf{u}_\theta \tag{3.336}$$

Now the acceleration of the particle in reference frame $\mathcal{F}$ is given as

$$^{\mathcal{F}}\mathbf{a} = \frac{{}^{\mathcal{F}}d}{dt}\left({}^{\mathcal{F}}\mathbf{v}\right) = \frac{{}^{\mathcal{F}}d}{dt}\left({}^{\mathcal{F}}\mathbf{v}_P\right) + \frac{{}^{\mathcal{F}}d}{dt}\left({}^{\mathcal{F}}\mathbf{v}_{m/P}\right) = {}^{\mathcal{F}}\mathbf{a}_P + {}^{\mathcal{F}}\mathbf{a}_{m/P} \tag{3.337}$$

Observing that the expression for ${}^{\mathcal{F}}\mathbf{v}_P$ as given in Eq. (3.332) is expressed in the basis $\{\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z\}$ and $\{\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z\}$ is fixed in reference frame $\mathcal{A}$, we can apply the rate of change transport theorem to ${}^{\mathcal{F}}\mathbf{v}_P$ between reference frames $\mathcal{A}$ and $\mathcal{F}$ to give

$$^{\mathcal{F}}\mathbf{a}_P = \frac{{}^{\mathcal{F}}d}{dt}\left({}^{\mathcal{F}}\mathbf{v}_P\right) = \frac{{}^{\mathcal{A}}d}{dt}\left({}^{\mathcal{F}}\mathbf{v}_P\right) + {}^{\mathcal{A}}\boldsymbol{\omega}^{\mathcal{F}} \times {}^{\mathcal{F}}\mathbf{v}_P \tag{3.338}$$

Now since R and ω are constant, we have that

$$\begin{aligned} \frac{{}^{\mathcal{A}}d}{dt}\left({}^{\mathcal{F}}\mathbf{v}_P\right) &= \mathbf{0} \\ {}^{\mathcal{F}}\boldsymbol{\omega}^{\mathcal{A}} \times {}^{\mathcal{F}}\mathbf{v}_P &= \omega\mathbf{e}_z \times R\omega\mathbf{e}_x = -R\omega^2\mathbf{e}_y \end{aligned} \tag{3.339}$$

Adding the two expressions in Eq. (3.339), we obtain the acceleration of point P in reference frame $\mathcal{F}$ as

$$^{\mathcal{F}}\mathbf{a}_P = -R\omega^2\mathbf{e}_x \tag{3.340}$$

Next, since ${}^{\mathcal{F}}\mathbf{v}_{m/P}$ is expressed in the basis $\{\mathbf{u}_r, \mathbf{u}_\theta, \mathbf{u}_z\}$ and $\{\mathbf{u}_r, \mathbf{u}_\theta, \mathbf{u}_z\}$ is fixed in reference frame $\mathcal{B}$, the acceleration of the particle relative to point P in reference frame $\mathcal{F}$ can be obtained by applying the rate of change transport theorem to ${}^{\mathcal{F}}\mathbf{v}_{m/P}$ between reference frames $\mathcal{B}$ and $\mathcal{F}$ as

$$^{\mathcal{F}}\mathbf{a}_{m/P} = \frac{{}^{\mathcal{F}}d}{dt}\left({}^{\mathcal{F}}\mathbf{v}_{m/P}\right) = \frac{{}^{\mathcal{B}}d}{dt}\left({}^{\mathcal{F}}\mathbf{v}_{m/P}\right) + {}^{\mathcal{F}}\boldsymbol{\omega}^{\mathcal{B}} \times {}^{\mathcal{F}}\mathbf{v}_{m/P} \tag{3.341}$$

Now we have that

$$\begin{aligned} \frac{{}^{\mathcal{B}}d}{dt}\left({}^{\mathcal{F}}\mathbf{v}_{m/P}\right) &= \ddot{r}\mathbf{u}_r + (\dot{r}\dot{\theta} + r\ddot{\theta})\mathbf{u}_\theta \\ {}^{\mathcal{F}}\boldsymbol{\omega}^{\mathcal{B}} \times {}^{\mathcal{F}}\mathbf{v}_{m/P} &= \dot{\theta}\mathbf{u}_z \times (\dot{r}\mathbf{u}_r + r\dot{\theta}\mathbf{u}_\theta) = -r\dot{\theta}^2\mathbf{u}_r + \dot{r}\dot{\theta}\mathbf{u}_\theta \end{aligned} \tag{3.342}$$

Adding the two expressions in Eq. (3.342), we obtain the acceleration of the particle relative to point P in reference frame $\mathcal{F}$ as

$$ {}^{\mathcal{F}}\mathbf{a}_{m/P} = (\ddot{r} - r\dot{\theta}^2)\mathbf{u}_r + (2\dot{r}\dot{\theta} + r\ddot{\theta})\mathbf{u}_\theta \tag{3.343} $$

Then, adding Eq. (3.340) and Eq. (3.343), we obtain the acceleration of the particle in reference frame $\mathcal{F}$ as

$$ {}^{\mathcal{F}}\mathbf{a} = -R\omega^2\mathbf{e}_x + (\ddot{r} - r\dot{\theta}^2)\mathbf{u}_r + (2\dot{r}\dot{\theta} + r\ddot{\theta})\mathbf{u}_\theta \tag{3.344} $$

Finally, using the expression for $\mathbf{e}_x$ in terms of $\{\mathbf{u}_r, \mathbf{u}_\theta\}$ from Eq. (3.322), the acceleration of the particle in reference frame $\mathcal{F}$ can be written in terms of the basis $\{\mathbf{u}_r, \mathbf{u}_\theta, \mathbf{u}_z\}$ as

$$ {}^{\mathcal{F}}\mathbf{a} = -R\omega^2[\cos(\theta-\omega t)\mathbf{u}_r - \sin(\theta-\omega t)\mathbf{u}_\theta] + (\ddot{r} - r\dot{\theta}^2)\mathbf{u}_r + (2\dot{r}\dot{\theta} + r\ddot{\theta})\mathbf{u}_\theta \tag{3.345} $$

Simplifying Eq. (3.345), we obtain

$$ {}^{\mathcal{F}}\mathbf{a} = [\ddot{r} - r\dot{\theta}^2 - R\omega^2\cos(\theta-\omega t)]\mathbf{u}_r + [2\dot{r}\dot{\theta} + r\ddot{\theta} + R\omega^2\sin(\theta-\omega t)]\mathbf{u}_\theta \tag{3.346} $$

Kinetics and Differential Equations of Motion

In order to obtain the two differential equation of motion for the particle, we need to apply Newton's 2^{nd} law, i.e., $\mathbf{F} = m\,{}^{\mathcal{F}}\mathbf{a}$. The free body diagram of the particle is shown in Fig. 3-11. It can be seen that the only force acting on the

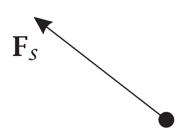

Figure 3-11 Free Body Diagram of Particle for Question 3–12 .

particle is due to the linear spring, $\mathbf{F}_s$. Consequently, we have that

$$ \mathbf{F}_s = -K\left[\ell - \ell_0\right]\mathbf{u}_s \tag{3.347} $$

Now we are given that the unstretched length of the spring is r_O which implies that $\ell_0 = r_0$. Furthermore, the attachment point of the spring is $\mathbf{r}_A = \mathbf{r}_P$. Consequently, the stretched length of the spring is given as

$$ \ell = \|\mathbf{r} - \mathbf{r}_A\| = \|\mathbf{r} - \mathbf{r}_P\| \tag{3.348} $$

Using the expression for $\mathbf{r}$ from Eq. (3.328) and the expression for $\mathbf{r}_P$ from Eq. (3.327), we obtain

$$ \ell = \|r\mathbf{u}_r + R\mathbf{e}_x - R\mathbf{e}_x\| = \|r\mathbf{u}_r\| = r \tag{3.349} $$

Finally, we have that

$$\mathbf{u}_s = \frac{\mathbf{r} - \mathbf{r}_A}{\|\mathbf{r} - \mathbf{r}_A\|} = \frac{\mathbf{r} - \mathbf{r}_P}{\|\mathbf{r} - \mathbf{r}_P\|} = \frac{r\mathbf{u}_r}{r} = \mathbf{u}_r \tag{3.350}$$

The spring force is then given as

$$\mathbf{F}_s = -K(r - r_0)\mathbf{u}_r \tag{3.351}$$

The resultant force acting on the particle is then given as

$$\mathbf{F} = \mathbf{F}_s = -K(r - r_0)\mathbf{u}_r \tag{3.352}$$

Then, setting $\mathbf{F}$ in Eq. (3.352) equal to $m{}^{\mathcal{F}}\mathbf{a}$ where ${}^{\mathcal{F}}\mathbf{a}$ is obtained from Eq. (3.346), we obtain

$$-K(r-r_0)\mathbf{u}_r = m[\ddot{r}-r\dot{\theta}^2-R\omega^2\cos(\theta-\omega t)]\mathbf{u}_r+m[2\dot{r}\dot{\theta}+r\ddot{\theta}+R\omega^2\sin(\theta-\omega t)]\mathbf{u}_\theta \tag{3.353}$$

Equating components in Eq. (3.353), we obtain the following two scalar equations:

$$m[\ddot{r} - r\dot{\theta}^2 - R\omega^2\cos(\theta - \omega t)] = -K(r - r_0) \tag{3.354}$$

$$m[2\dot{r}\dot{\theta} + r\ddot{\theta} + R\omega^2\sin(\theta - \omega t)] = 0 \tag{3.355}$$

It can be seen that neither Eq. (3.354) nor Eq. (3.355) contains any unknown reactions forces. Consequently, Eq. (3.354) and Eq. (3.355) are the two differential equations of motion for the particle. We can re-write Eq. (3.354) and Eq. (3.355) in a slightly different form to give

$$m[\ddot{r} - r\dot{\theta}^2 - R\omega^2\cos(\theta - \omega t)] + K(r - r_0) = 0 \tag{3.356}$$

$$m[2\dot{r}\dot{\theta} + r\ddot{\theta} + R\omega^2\sin(\theta - \omega t)] = 0 \tag{3.357}$$

Question 3–13

A particle of mass m slides without friction along a surface in form of a paraboloid as shown in Fig. P3-13. The equation for the paraboloid is

$$z = \frac{r^2}{2R}$$

where z is the height of the particle above the horizontal plane, r is the distance from O to Q where Q is the projection of P onto the horizontal plane, and R is a constant. Knowing that θ is the angle formed by the direction OQ with the x-axis and that gravity acts downward, determine a system of two differential equations.

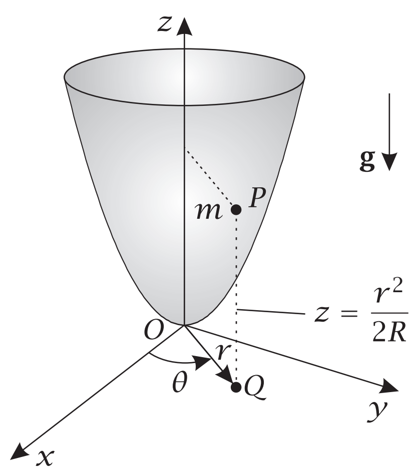

Figure P3-13

Solution to Question 3–13

Kinematics

First, let $\mathcal{F}$ be a reference frame fixed to the paraboloid. Then, choose the following coordinate system fixed in reference frame $\mathcal{F}$:

	Origin at O	
$\mathbf{E}_x$	=	Along Ox
$\mathbf{E}_y$	=	Along Oy
$\mathbf{E}_z$	=	$\mathbf{E}_x \times \mathbf{E}_y$

Next, let $\mathcal{A}$ be a reference frame fixed to the plane formed by the vectors $\mathbf{E}_z$ and OQ. Then, choose the following coordinate system fixed in reference frame $\mathcal{A}$:

$$\begin{array}{ccc} & \text{Origin at } O & \\ \mathbf{e}_r & = & \text{Along } OQ \\ \mathbf{E}_z & = & \text{Up} \\ \mathbf{e}_\theta & = & \mathbf{E}_z \times \mathbf{e}_r \end{array}$$

The position of the particle is then given as

$$\mathbf{r} = r\mathbf{e}_r + \frac{r^2}{2R}\mathbf{E}_z \tag{3.358}$$

Furthermore, the angular velocity of reference frame $\mathcal{A}$ in reference frame $\mathcal{F}$ is given as

$${}^{\mathcal{F}}\boldsymbol{\omega}^{\mathcal{A}} = \dot{\theta}\mathbf{E}_z \tag{3.359}$$

The velocity of the particle in reference frame $\mathcal{F}$ is then obtained from the rate of change transport theorem as

$${}^{\mathcal{F}}\mathbf{v} = \frac{{}^{\mathcal{F}}d\mathbf{r}}{dt} = \frac{{}^{\mathcal{A}}d\mathbf{r}}{dt} + {}^{\mathcal{F}}\boldsymbol{\omega}^{\mathcal{A}} \times \mathbf{r} \tag{3.360}$$

Now we note that

$$\begin{aligned} \frac{{}^{\mathcal{A}}d\mathbf{r}}{dt} &= \dot{r}\mathbf{e}_r + \frac{r\dot{r}}{R}\mathbf{E}_z \\ {}^{\mathcal{F}}\boldsymbol{\omega}^{\mathcal{A}} \times \mathbf{r} &= \dot{\theta}\mathbf{E}_z \times \left(r\mathbf{e}_r + \frac{r^2}{2R}\mathbf{E}_z\right) = r\dot{\theta}\mathbf{e}_\theta \end{aligned} \tag{3.361}$$

Adding the two expressions in Eq. (3.361), we obtain the velocity of the particle in reference frame $\mathcal{F}$ as

$${}^{\mathcal{F}}\mathbf{v} = \dot{r}\mathbf{e}_r + r\dot{\theta}\mathbf{e}_\theta + \frac{r\dot{r}}{R}\mathbf{E}_z \tag{3.362}$$

Next, applying the rate of change transport theorem to ${}^{\mathcal{F}}\mathbf{v}$, we obtain the acceleration of the particle in reference frame $\mathcal{F}$ as

$${}^{\mathcal{F}}\mathbf{a} = \frac{{}^{\mathcal{F}}d}{dt}\left({}^{\mathcal{F}}\mathbf{v}\right) = \frac{{}^{\mathcal{A}}d}{dt}\left({}^{\mathcal{F}}\mathbf{v}\right) + {}^{\mathcal{F}}\boldsymbol{\omega}^{\mathcal{A}} \times {}^{\mathcal{F}}\mathbf{v} \tag{3.363}$$

Now we have that

$$\begin{aligned} \frac{{}^{\mathcal{A}}d}{dt}\left({}^{\mathcal{F}}\mathbf{v}\right) &= \ddot{r}\mathbf{e}_r + (\dot{r}\dot{\theta} + r\ddot{\theta})\mathbf{e}_\theta + \frac{\dot{r}^2 + r\ddot{r}}{R}\mathbf{E}_z \\ {}^{\mathcal{F}}\boldsymbol{\omega}^{\mathcal{A}} \times {}^{\mathcal{F}}\mathbf{v} &= \dot{\theta}\mathbf{E}_z \times \left(\dot{r}\mathbf{e}_r + r\dot{\theta}\mathbf{e}_\theta + \frac{r\dot{r}}{R}\mathbf{E}_z\right) = \dot{r}\dot{\theta}\mathbf{e}_\theta - r\dot{\theta}^2\mathbf{e}_r \end{aligned} \tag{3.364}$$

Adding the expressions in Eq. (3.364), we obtain the acceleration of the particle in reference frame $\mathcal{F}$ as

$${}^{\mathcal{F}}\mathbf{a} = (\ddot{r} - r\dot{\theta}^2)\mathbf{e}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\mathbf{e}_\theta + \frac{\dot{r}^2 + r\ddot{r}}{R}\mathbf{E}_z \tag{3.365}$$

Kinetics

We need to apply Newton's 2^{nd} Law, i.e. $\mathbf{F} = m^{\mathcal{F}}\mathbf{a}$. The free body diagram of the particle is shown in Fig. 3-12. We note from Fig. 3-12 that $\mathbf{N}$ is the reaction

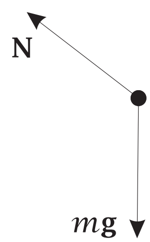

Figure 3-12 Free Body Diagram of Particle for Question 3–13.

force of the paraboloid on the particle and that $m\mathbf{g}$ is the force due to gravity. We further note that $\mathbf{N}$ must lie *normal* to the surface at the point of contact. Now we note that the vector that is normal to a surface is in the direction of the gradient of the function that defines the surface. In order to compute the gradient, we re-write the equation for the paraboloid in the following form:

$$f(r,\theta,z) = z - \frac{r^2}{2R} = 0 \tag{3.366}$$

Then, from calculus, the gradient in obtained in cylindrical coordinates as

$$\nabla f = \frac{\partial f}{\partial r}\mathbf{e}_r + \frac{1}{r}\frac{\partial f}{\partial \theta}\mathbf{e}_\theta + \frac{\partial f}{\partial z}\mathbf{E}_z \tag{3.367}$$

Computing the gradient, we obtain

$$\nabla f = -\frac{r}{R}\mathbf{e}_r + \mathbf{E}_z \tag{3.368}$$

The unit vector in the direction of the gradient is then given as

$$\mathbf{e}_n = \frac{\nabla f}{\|\nabla f\|} = \frac{-\frac{r}{R}\mathbf{e}_r + \mathbf{E}_z}{\sqrt{1 + \left(\frac{r}{R}\right)^2}} \tag{3.369}$$

which implies that the normal force is

$$\mathbf{N} = N\mathbf{e}_n = N\left[\frac{-\frac{r}{R}\mathbf{e}_r + \mathbf{E}_z}{\sqrt{1 + \left(\frac{r}{R}\right)^2}}\right] \tag{3.370}$$

Next, the force of gravity is

$$m\mathbf{g} = -mg\mathbf{E}_z \tag{3.371}$$

The resultant force on the particle is then given as

$$\mathbf{F} = \mathbf{N} + m\mathbf{g} = N\left[\frac{-\dfrac{r}{R}\mathbf{e}_r + \mathbf{E}_z}{\sqrt{1+\left(\dfrac{r}{R}\right)^2}}\right] - mg\mathbf{E}_z \tag{3.372}$$

which can be re-written as

$$\mathbf{F} = -N\frac{\dfrac{r}{R}}{\sqrt{1+\left(\dfrac{r}{R}\right)^2}}\mathbf{e}_r + \left(N\frac{1}{\sqrt{1+\left(\dfrac{r}{R}\right)^2}} - mg\right)\mathbf{E}_z \tag{3.373}$$

Setting $\mathbf{F} = m{}^{\mathcal{F}}\mathbf{a}$ using ${}^{\mathcal{F}}\mathbf{a}$ from part (a), we obtain

$$-N\frac{\dfrac{r}{R}}{\sqrt{1+\left(\dfrac{r}{R}\right)^2}}\mathbf{e}_r + \left(N\frac{1}{\sqrt{1+\left(\dfrac{r}{R}\right)^2}} - mg\right)\mathbf{E}_z = m(\ddot{r}-r\dot{\theta}^2)\mathbf{e}_r + m(r\ddot{\theta}+2\dot{r}\dot{\theta})\mathbf{e}_\theta + m\frac{\dot{r}^2+r\ddot{r}}{R}\mathbf{E}_z \tag{3.374}$$

We then obtain the following three scalar equations:

$$\begin{aligned} -N\frac{\dfrac{r}{R}}{\sqrt{1+\left(\dfrac{r}{R}\right)^2}} &= m(\ddot{r}-r\dot{\theta}^2) \\ 0 &= m(r\ddot{\theta}+2\dot{r}\dot{\theta}) \\ \left(N\frac{1}{\sqrt{1+\left(\dfrac{r}{R}\right)^2}} - mg\right) &= m\frac{\dot{r}^2+r\ddot{r}}{R} \end{aligned} \tag{3.375}$$

A system of two differential equations can then be obtained as follows. The first differential equation is simply the second equation in Eq. (3.375), i.e.

$$m(r\ddot{\theta}+2\dot{r}\dot{\theta}) = 0 \tag{3.376}$$

Dropping m, we obtain

$$r\ddot{\theta}+2\dot{r}\dot{\theta} = 0 \tag{3.377}$$

Next, rearranging the third equation in Eq. (3.375) by adding mg to both sides, we obtain

$$N\frac{1}{\sqrt{1+\left(\dfrac{r}{R}\right)^2}} = mg + m\frac{\dot{r}^2+r\ddot{r}}{R} \tag{3.378}$$

Then, dividing the first equation in Eq. (3.375) by this last result, we obtain

$$-\frac{r}{R} = \frac{\ddot{r} - r\dot{\theta}^2}{\dfrac{\dot{r}^2 + r\ddot{r}}{R} + g} \tag{3.379}$$

Rearranging and simplifying this last equation, we obtain the second differential equation as

$$\left[1 + \left(\frac{r}{R}\right)^2\right]\ddot{r} + \frac{r}{R^2}\dot{r}^2 - r\dot{\theta}^2 + \frac{gr}{R} = 0 \tag{3.380}$$

The system of two differential equations is then given as

$$\begin{array}{rcl} r\ddot{\theta} + 2\dot{r}\dot{\theta} & = & 0 \\ \left[1 + \left(\frac{r}{R}\right)^2\right]\ddot{r} + \frac{r}{R^2}\dot{r}^2 - r\dot{\theta}^2 + \frac{gr}{R} & = & 0 \end{array} \tag{3.381}$$

Conservation of Energy

Two forces act on the particle: **N** and $m\mathbf{g}$. We know that the force of gravity is conservative, but we do not know anything about **N**. However, we note the following about **N**:

$$\mathbf{N} \cdot {}^{\mathcal{F}}\mathbf{v} = \mathbf{N} = N\left[\frac{-\dfrac{r}{R}\mathbf{e}_r + \mathbf{E}_z}{\sqrt{1 + \left(\dfrac{r}{R}\right)^2}}\right] \cdot \left[\dot{r}\mathbf{e}_r + r\dot{\theta}\mathbf{e}_\theta + \frac{r\dot{r}}{R}\mathbf{E}_z\right] \tag{3.382}$$

Simplifying Eq. (3.382), we obtain

$$\mathbf{N} \cdot {}^{\mathcal{F}}\mathbf{v} = N\left\{\frac{-\dfrac{r}{R}}{\sqrt{1 + \left(\dfrac{r}{R}\right)^2}}\dot{r} + \frac{\dfrac{r}{R}}{\sqrt{1 + \left(\dfrac{r}{R}\right)^2}}\dot{r}\right\} = 0 \tag{3.383}$$

Therefore, **N** does no work. Consequently, from the work-energy theorem we have that

$$\frac{d}{dt}\left({}^{\mathcal{F}}E\right) = \mathbf{N} \cdot \mathbf{v} = 0 \tag{3.384}$$

which implies that ${}^{\mathcal{F}}E$ = constant, i.e. energy is conserved.

Question 3–17

A particle of mass m is attached to an inextensible massless rope of length l as shown in Fig. P3-17. The rope is attached at its other end to point A located at the top of a fixed cylinder of radius R. As the particle moves, the rope wraps itself around the cylinder and never becomes slack. Knowing that θ is the angle measured from the vertical to the point of tangency of the exposed portion of the rope with the cylinder and that gravity acts downward, determine the differential equation of motion for the particle in terms of the angle θ. You may assume in your solution that the angle θ is always positive.

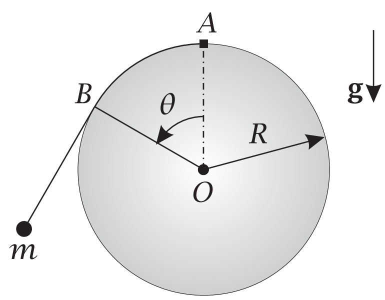

Figure P3-17

Solution to Question 3–17

Kinematics

First, let $\mathcal{F}$ be a reference frame fixed to the circular track. Then, choose the following coordinate system fixed in reference frame $\mathcal{F}$:

$$\begin{array}{ccc} & \text{Origin at } O & \\ \mathbf{E}_x & = & \text{Along } OA \\ \mathbf{E}_z & = & \text{Out of Page} \\ \mathbf{E}_y & = & \mathbf{E}_z \times \mathbf{E}_x \end{array}$$

Next, let $\mathcal{A}$ be a reference frame fixed to the exposed portion of the rope. Then, choose the following coordinate system fixed in reference frame $\mathcal{A}$:

$$\begin{array}{ccc} & \text{Origin at } O & \\ \mathbf{e}_r & = & \text{Along } OB \\ \mathbf{e}_z & = & \text{Out of Page} \\ \mathbf{e}_\theta & = & \mathbf{e}_z \times \mathbf{e}_r \end{array}$$

The geometry of the bases $\{\mathbf{E}_x, \mathbf{E}_y, \mathbf{E}_z\}$ and $\{\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_z\}$ is shown in Fig. 3-13. Using Fig. 3-13, we have that

$$\mathbf{E}_x = \cos\theta\,\mathbf{e}_r - \sin\theta\,\mathbf{e}_\theta \tag{3.385}$$

$$\mathbf{E}_y = \sin\theta\,\mathbf{e}_r + \cos\theta\,\mathbf{e}_\theta \tag{3.386}$$

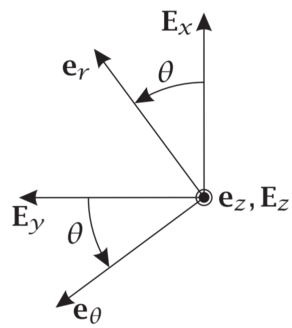

Figure 3-13 Geometry of Question 3–17.

Now we note that the rope has a fixed length l. Since the length of the portion of the rope wrapped around the cylinder is $R\theta$, the exposed portion of the rope must have length $l - R\theta$. Furthermore, since the exposed portion of the rope lies along the direction from B to m, the position of the particle is given as

$$\mathbf{r} = R\mathbf{e}_r + (l - R\theta)\mathbf{e}_\theta \tag{3.387}$$

Furthermore, since the direction along OB is fixed to reference frame $\mathcal{A}$, the angular velocity of reference frame $\mathcal{A}$ in reference frame $\mathcal{F}$ is given as

$$^{\mathcal{F}}\boldsymbol{\omega}^{\mathcal{A}} = \dot{\theta}\mathbf{e}_z \tag{3.388}$$

The velocity of the particle is then computed using the rate of change transport theorem as

$$^{\mathcal{F}}\mathbf{v} = \frac{^{\mathcal{F}}d\mathbf{r}}{dt} = \frac{^{\mathcal{A}}d\mathbf{r}}{dt} + {}^{\mathcal{F}}\boldsymbol{\omega}^{\mathcal{A}} \times \mathbf{r} \tag{3.389}$$

Now we have that

$$\frac{^{\mathcal{A}}d\mathbf{r}}{dt} = -R\dot{\theta}\mathbf{e}_\theta \tag{3.390}$$

$$^{\mathcal{F}}\boldsymbol{\omega}^{\mathcal{A}} \times \mathbf{r} = \dot{\theta}\mathbf{e}_z \times [R\mathbf{e}_r + (l - R\theta)\mathbf{e}_\theta] = R\dot{\theta}\mathbf{e}_\theta - (l - R\theta)\dot{\theta}\mathbf{e}_r \tag{3.391}$$

Adding Eq. (3.390) and Eq. (3.391), we obtain the velocity of the particle in reference frame $\mathcal{F}$ as

$$^{\mathcal{F}}\mathbf{v} = -(l - R\theta)\dot{\theta}\mathbf{e}_r \tag{3.392}$$

The acceleration of the particle in reference frame $\mathcal{F}$ is then obtained by applying the rate of change transport theorem to $^{\mathcal{F}}\mathbf{v}$ as

$$^{\mathcal{F}}\mathbf{a} = \frac{^{\mathcal{F}}d}{dt}\left(^{\mathcal{F}}\mathbf{v}\right) = \frac{^{\mathcal{A}}d}{dt}\left(^{\mathcal{F}}\mathbf{v}\right) + {}^{\mathcal{F}}\boldsymbol{\omega}^{\mathcal{A}} \times {}^{\mathcal{F}}\mathbf{v} \tag{3.393}$$

Now we have that

$$\frac{{}^{\mathcal{A}}d}{dt}\left({}^{\mathcal{F}}\mathbf{v}\right) = -\left[(-R\dot{\theta})\dot{\theta} + (l - R\theta)\ddot{\theta}\right]\mathbf{e}_r \tag{3.394}$$

$${}^{\mathcal{F}}\boldsymbol{\omega}^{\mathcal{A}} \times {}^{\mathcal{F}}\mathbf{v} = \dot{\theta}\mathbf{e}_z \times \left[-(l - R\theta)\dot{\theta}\mathbf{e}_r\right] = -(l - R\theta)\dot{\theta}^2\mathbf{e}_\theta \tag{3.395}$$

Adding Eq. (3.394) and Eq. (3.395), we obtain

$${}^{\mathcal{F}}\mathbf{a} = -\left[-R\dot{\theta}^2 + (l - R\theta)\ddot{\theta}\right]\mathbf{e}_r - (l - R\theta)\dot{\theta}^2\mathbf{e}_\theta \tag{3.396}$$

Eq. (3.396) simplifies to

$${}^{\mathcal{F}}\mathbf{a} = \left[R\dot{\theta}^2 - (l - R\theta)\ddot{\theta}\right]\mathbf{e}_r - (l - R\theta)\dot{\theta}^2\mathbf{e}_\theta \tag{3.397}$$

Kinetics

The free body diagram of the particle is shown in Fig. 3-14. From Fig. 3-14 it can

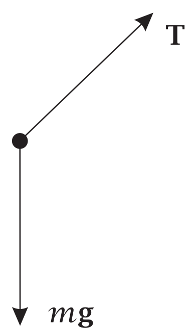

Figure 3-14 Free Body Diagram for Question 3–17.

be seen that the two forces acting on the particle are

$$\begin{aligned} \mathbf{T} &= \text{Tension in Rope} \\ m\mathbf{g} &= \text{Force of Gravity} \end{aligned}$$

Since the tension must act along the direction of the exposed portion of the rope and gravity acts vertically downward, we have that

$$\mathbf{T} = T\mathbf{e}_\theta \tag{3.398}$$

$$m\mathbf{g} = -mg\mathbf{E}_x \tag{3.399}$$

Then, using the expresion for $\mathbf{E}_x$ from Eq. (3.385), the force of gravity can be expressed in the basis $\{\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_z\}$ as

$$m\mathbf{g} - mg(\cos\theta\,\mathbf{e}_r - \sin\theta\,\mathbf{e}_\theta) = -mg\cos\theta\,\mathbf{e}_r + mg\sin\theta\,\mathbf{e}_\theta \tag{3.400}$$

The resultant force on the particle is then given as

$$\mathbf{F} = \mathbf{T} + m\mathbf{g} = T\mathbf{e}_\theta - mg\cos\theta\,\mathbf{e}_r + mg\sin\theta\,\mathbf{e}_\theta = -mg\cos\theta\,\mathbf{e}_r + (T + mg\sin\theta)\mathbf{e}_\theta \tag{3.401}$$

Determination of Differential Equation Using Newton's 2^{nd} Law

Setting $\mathbf{F}$ from Eq. (3.401) equal to $m^{\mathcal{F}}\mathbf{a}$ using ${}^{\mathcal{F}}\mathbf{a}$ from Eq. (3.397), we have that

$$-mg\cos\theta\,\mathbf{e}_r + (T + mg\sin\theta)\mathbf{e}_\theta = m\left[R\dot{\theta}^2 - (l - R\theta)\ddot{\theta}\right]\mathbf{e}_r - m(l - R\theta)\dot{\theta}^2\mathbf{e}_\theta \tag{3.402}$$

Equating components in Eq. (3.402) results in the following two scalar equations:

$$m\left[R\dot{\theta}^2 - (l - R\theta)\ddot{\theta}\right] = -mg\cos\theta \tag{3.403}$$

$$m(l - R\theta)\dot{\theta}^2 = T + mg\sin\theta \tag{3.404}$$

Then, since Eq. (3.403) has no unknown reaction forces, it is the differential equation, i.e., the differential equation of motion is given as

$$m\left[R\dot{\theta}^2 - (l - R\theta)\ddot{\theta}\right] = -mg\cos\theta \tag{3.405}$$

Simplifying Eq. (3.405) by dropping m and rearranging, we obtain the differential equation as

$$(l - R\theta)\ddot{\theta} - R\dot{\theta}^2 - g\cos\theta = 0 \tag{3.406}$$

Determination of Differential Equation Using Alternate Form of Work-Energy Theorem

Since the motion of the particle can be described using a single variable (namely, θ), we can apply the work-energy theorem for a particle to obtain the differential equation of motion. In particular, we will use the alternate form of the work-energy theorem for a particle in reference frame $\mathcal{F}$ as

$$\frac{d}{dt}\left({}^{\mathcal{F}}E\right) = \mathbf{F}_{nc} \cdot {}^{\mathcal{F}}\mathbf{v} \tag{3.407}$$

Now the total energy in reference frame $\mathcal{F}$ is given as

$${}^{\mathcal{F}}E = {}^{\mathcal{F}}T + {}^{\mathcal{F}}U \tag{3.408}$$

First, we have the kinetic energy in reference frame $\mathcal{F}$ as

$${}^{\mathcal{F}}T = \frac{1}{2}m^{\mathcal{F}}\mathbf{v} \cdot {}^{\mathcal{F}}\mathbf{v} \tag{3.409}$$

Substituting ${}^{\mathcal{F}}\mathbf{v}$ from Eq. (3.392), we have that

$${}^{\mathcal{F}}T = \frac{1}{2}m(l - R\theta)\dot{\theta}\mathbf{e}_\theta \cdot (l - R\theta)\dot{\theta}\mathbf{e}_\theta \tag{3.410}$$

Eq. (3.410) simplifies to

$${}^{\mathcal{F}}T = \frac{1}{2}m(l - R\theta)^2\dot{\theta}^2 \tag{3.411}$$

Furthermore, since the only conservative force acting on the particle is due to gravity and gravity is a constant force, the potential energy is given as

$$^{\mathcal{F}}U = {}^{\mathcal{F}}U_g = -m\mathbf{g}\cdot\mathbf{r} \tag{3.412}$$

Substituting $m\mathbf{g}$ from Eq. (3.400) and $\mathbf{r}$ from Eq. (3.387), we obtain

$$\begin{aligned} ^{\mathcal{F}}U &= -(-mg\cos\theta\,\mathbf{e}_r + mg\sin\theta\,\mathbf{e}_\theta)\cdot(R\mathbf{e}_r + (l-R\theta)\mathbf{e}_\theta) \\ &= mgR\cos\theta - mg(l-R\theta)\sin\theta \end{aligned} \tag{3.413}$$

The total energy of the system is then obtained by adding T from Eq. (3.411) and U from Eq. (3.413) as

$$^{\mathcal{F}}E = {}^{\mathcal{F}}T + {}^{\mathcal{F}}U = \frac{1}{2}m(l-R\theta)^2\dot{\theta}^2 + mgR\cos\theta - mg(l-R\theta)\sin\theta \tag{3.414}$$

Next, the only force *other* than gravity acting on the particle is that due to the tension in the rope. Using the expression for the tension in the rope from Eq. (3.398) and the velocity of the particle from Eq. (3.392), the power of the tension force is given as

$$\mathbf{T}\cdot{}^{\mathcal{F}}\mathbf{v} = T\mathbf{e}_\theta\cdot(l-R\theta)\dot{\theta}\mathbf{e}_r = 0 \tag{3.415}$$

Consequently, the power produced by all non-conservative forces is zero which implies that

$$\frac{d}{dt}\left({}^{\mathcal{F}}E\right) = 0 \tag{3.416}$$

Differentiating $^{\mathcal{F}}E$ from Eq. (3.414) and setting the result equal to zero, we obtain

$$\begin{aligned} \frac{d}{dt}\left({}^{\mathcal{F}}E\right) &= m(l-R\theta)(-R\dot{\theta})\dot{\theta}^2 + m(l-R\theta)^2\dot{\theta}\ddot{\theta} - mgR\dot{\theta}\sin\theta \\ &\quad + mgR\dot{\theta}\sin\theta - mg(l-R\theta)\dot{\theta}\cos\theta = 0 \end{aligned} \tag{3.417}$$

Factoring out m and $\dot{\theta}$ from Eq. (3.417), we obtain

$$m\dot{\theta}\left[-(l-R\theta)^2\ddot{\theta} - R(l-R\theta)\dot{\theta}^2 - g(l-R\theta)\cos\theta\right] = 0 \tag{3.418}$$

Noting that $\dot{\theta} \neq 0$, we can drop m and $\dot{\theta}$ from Eq. (3.418) to give

$$(l-R\theta)\ddot{\theta} - R\dot{\theta}^2 - g\cos\theta = 0 \tag{3.419}$$

It can be seen that Eq. (3.419) is identical the result obtained using Newton's 2^{nd} law as shown in Eq. (3.406).

Question 3–19

A collar of mass m slides without friction along a circular track of radius R as shown in Fig. P3-19. Attached to the collar is a linear spring with spring constant K and unstretched length zero. The spring is attached at the fixed point A located a distance $2R$ from the center of the circle. Assuming no gravity and the initial conditions $\theta(t=0)=\theta_0$ and $\dot{\theta}(t=0)=\dot{\theta}_0$, determine (a) the differential equation of motion for the collar in terms the angle θ and (b) the reaction force exerted by the track on the collar as a function of the angle θ.

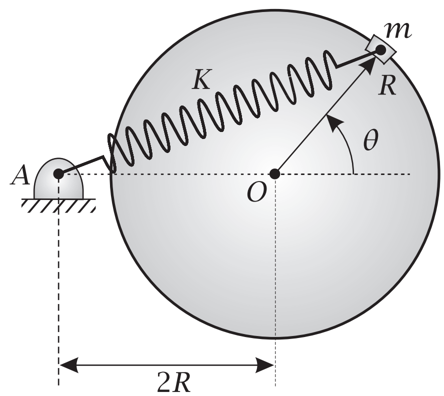

Figure P3-19

Solution to Question 3–19

Kinematics

First, let $\mathcal{F}$ be a reference frame fixed to the track. Then, choose the following coordinate system fixed in reference frame $\mathcal{F}$:

	Origin at O	
$\mathbf{E}_x$	$=$	Along Om at $t=0$
$\mathbf{E}_z$	$=$	Out of Page
$\mathbf{E}_y$	$=$	$\mathbf{E}_z \times \mathbf{E}_x$

Next, let $\mathcal{A}$ be a reference frame fixed to the direction of Om. Then, choose the following coordinate system fixed in reference frame $\mathcal{A}$:

	Origin at O	
$\mathbf{e}_r$	$=$	Along Om
$\mathbf{e}_z$	$=$	Out of Page
$\mathbf{e}_\theta$	$=$	$\mathbf{E}_z \times \mathbf{e}_r$

The geometry of the bases $\{\mathbf{E}_x, \mathbf{E}_y, \mathbf{E}_z\}$ and $\{\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_z\}$ is shown in Fig. 3-15 from which we obtain

$$\begin{aligned} \mathbf{E}_x &= \cos\theta\,\mathbf{e}_r - \sin\theta\,\mathbf{e}_\theta & (3.420) \\ \mathbf{E}_y &= \sin\theta\,\mathbf{e}_r + \cos\theta\,\mathbf{e}_\theta & (3.421) \end{aligned}$$

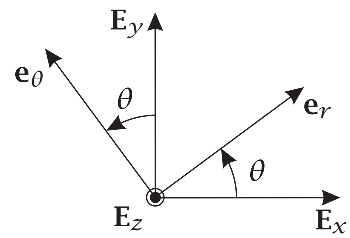

Figure 3-15 Geometry of Question .

The position of the particle is then given as

$$\mathbf{r} = R\mathbf{e}_r \tag{3.422}$$

Furthermore, the angular velocity of reference frame $\mathcal{A}$ in reference frame $\mathcal{F}$ is given as

$$^{\mathcal{F}}\boldsymbol{\omega}^{\mathcal{A}} = \dot{\theta}\mathbf{e}_z \tag{3.423}$$

The velocity of the particle in reference frame $\mathcal{F}$ is then obtained from the rate of change transport theorem as

$$^{\mathcal{F}}\mathbf{v} = \frac{^{\mathcal{F}}d\mathbf{r}}{dt} = \frac{^{\mathcal{A}}d\mathbf{r}}{dt} + {^{\mathcal{F}}\boldsymbol{\omega}^{\mathcal{A}}} \times \mathbf{r} \tag{3.424}$$

Now we have that

$$\begin{aligned} \frac{^{\mathcal{A}}d\mathbf{r}}{dt} &= \mathbf{0} & (3.425) \\ {^{\mathcal{F}}\boldsymbol{\omega}^{\mathcal{A}}} \times \mathbf{r} &= \dot{\theta}\mathbf{e}_z \times R\mathbf{e}_r = R\dot{\theta}\mathbf{e}_\theta & (3.426) \end{aligned}$$

Adding the expressions in Eq. (3.425) and Eq. (3.426), we obtain the velocity of the collar in reference frame $\mathcal{F}$ as

$$^{\mathcal{F}}\mathbf{v} = R\dot{\theta}\mathbf{e}_\theta \tag{3.427}$$

The acceleration of the collar in reference frame $\mathcal{F}$ is obtained by applying the rate of change transport theorem to $^{\mathcal{F}}\mathbf{v}$ as

$$^{\mathcal{F}}\mathbf{a} = \frac{^{\mathcal{F}}d}{dt}\left({^{\mathcal{F}}\mathbf{v}}\right) + {^{\mathcal{F}}\boldsymbol{\omega}^{\mathcal{A}}} \times {^{\mathcal{F}}\mathbf{v}} \tag{3.428}$$

Now we have that

$$\frac{^{\mathcal{F}}d}{dt}\left(^{\mathcal{F}}\mathbf{v}\right) = R\ddot{\theta}\mathbf{e}_\theta \tag{3.429}$$

$$^{\mathcal{F}}\boldsymbol{\omega}^{\mathcal{A}} \times {}^{\mathcal{F}}\mathbf{v} = \dot{\theta}\mathbf{e}_z \times (R\dot{\theta}\mathbf{e}_\theta) = -R\dot{\theta}^2\mathbf{e}_r \tag{3.430}$$

Adding the expressions in Eq. (3.429) and Eq. (3.430), we obtain the acceleration of the collar in reference frame $\mathcal{F}$ as

$$^{\mathcal{F}}\mathbf{v} = -R\dot{\theta}^2\mathbf{e}_r + R\ddot{\theta}\mathbf{e}_\theta \tag{3.431}$$

Kinetics

The differential equation of motion will be determined by applying Newton's 2^{nd} law. First, the free body diagram of the collar is shown in Fig. 3-16. Using

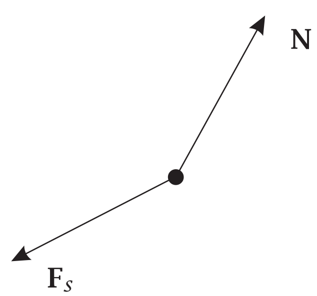

Figure 3-16 Free Body Diagram for Question 3–19.

Fig. 3-16, we see that the following two forces act on the collar:

$\mathbf{N}$ = Reaction force of Track on Particle
$\mathbf{F}_s$ = Spring Force

Now we know that the reaction force $\mathbf{N}$ is orthogonal to the track. Furthermore, since the collar is undergoing circular motion, the tangent vector to the track is in the direction $\mathbf{e}_\theta$. Consequently, the direction normal to the track is $\mathbf{e}_r$. Therefore, the reaction force can be written as

$$\mathbf{N} = N\mathbf{e}_r \tag{3.432}$$

Next, the general expression for a spring force is

$$\mathbf{F}_s = -K(\ell - \ell_0)\mathbf{u}_s \tag{3.433}$$

Now we recall for a spring that

$$\ell = \|\mathbf{r} - \mathbf{r}_A\| \tag{3.434}$$

$$\mathbf{u}_s = \frac{\mathbf{r} - \mathbf{r}_A}{\|\mathbf{r} - \mathbf{r}_A\|} \tag{3.435}$$

where A is the attachment point of the spring. It is seen from the geometry that the spring is attached a distance $2a$ from the center of the circle. In terms of the direction $\mathbf{E}_x$, we have that

$$\mathbf{r}_A = -2R\mathbf{E}_x \tag{3.436}$$

Then, using the expression for $\mathbf{r}$ from Eq. (3.422), we obtain

$$\mathbf{r} - \mathbf{r}_A = R\mathbf{e}_r + 2R\mathbf{E}_x \tag{3.437}$$

Therefore, the stretched length of the spring is obtained as

$$\ell = \|\mathbf{r} - \mathbf{r}_A\| = \|R\mathbf{e}_r + 2R\mathbf{E}_x\| \tag{3.438}$$

Then, substituting the result of Eq. (3.434) into Eq. (3.435), we obtain

$$\mathbf{u}_s = \frac{\mathbf{r} - \mathbf{r}_A}{\|\mathbf{r} - \mathbf{r}_A\|} = \frac{R\mathbf{e}_r + 2R\mathbf{E}_x}{\|R\mathbf{e}_r + 2R\mathbf{E}_x\|} \tag{3.439}$$

Substituting the results of Eq. (3.438) and Eq. (3.439) into Eq. (3.433) and using the fact that $\ell_0 = 0$, we have that

$$\mathbf{F}_s = -K\|R\mathbf{e}_r + 2R\mathbf{E}_x\| \frac{R\mathbf{e}_r + 2R\mathbf{E}_x}{\|R\mathbf{e}_r + 2R\mathbf{E}_x\|} = -K(R\mathbf{e}_r + 2R\mathbf{E}_x) = -KR(\mathbf{e}_r + 2\mathbf{E}_x) \tag{3.440}$$

Then, substituting the expression for $\mathbf{E}_x$ from Eq. (3.420) into Eq. (3.440), we obtain

$$\mathbf{F}_s = -KR\mathbf{e}_r - 2KR(\cos\theta\,\mathbf{e}_r - \sin\theta\,\mathbf{e}_\theta) = -KR(1 + 2\cos\theta)\mathbf{e}_r + 2KR\sin\theta\,\mathbf{e}_\theta \tag{3.441}$$

The resultant force acting on the particle is then given as

$$\mathbf{F} = \mathbf{N} + \mathbf{F}_s = [N - KR(1 + 2\cos\theta)]\,\mathbf{e}_r + 2KR\sin\theta\,\mathbf{e}_\theta \tag{3.442}$$

Setting $\mathbf{F}$ in Eq. (3.442) equal to $m\,{}^{\mathcal{F}}\mathbf{a}$ using ${}^{\mathcal{F}}\mathbf{a}$ from Eq. (3.431), we obtain

$$[N - KR(1 + 2\cos\theta)]\,\mathbf{e}_r + 2KR\sin\theta\,\mathbf{e}_\theta = -mR\dot{\theta}^2 + mR\ddot{\theta}\mathbf{e}_\theta \tag{3.443}$$

We then obtain the following two scalar equations:

$$-mR\dot{\theta}^2 = N - KR(1 + 2\cos\theta) \tag{3.444}$$

$$mR\ddot{\theta} = 2KR\sin\theta \tag{3.445}$$

(a) Differential Equation of Motion

Since Eq. (3.445) has no reaction forces and the motion of the particle is described using a single variable (i.e., θ), the differential equation of motion is given as

$$mR\ddot{\theta} = 2KR\sin\theta \tag{3.446}$$

Simplifying Eq. (3.446), we obtain

$$\ddot{\theta} - \frac{2K}{m}\sin\theta = 0 \tag{3.447}$$

(b) Reaction Force of Track on Particle As a Function of θ

Solving for the reaction force using Eq. (3.444), we obtain

$$N = -mR\dot{\theta}^2 + KR(1 + 2\cos\theta) \tag{3.448}$$

It is seen from Eq. (3.448) that, in order to obtain N as a function of θ, it is necessary to find $\dot{\theta}^2$ as a function of θ. We can obtain $\dot{\theta}^2$ in terms of θ using the differential equation in Eq. (3.447). First, we have from the chain rule that

$$\ddot{\theta} = \dot{\theta}\frac{d\dot{\theta}}{d\theta} = \frac{2K}{m}\sin\theta \tag{3.449}$$

Separating variables in Eq. (3.449), we obtain

$$\dot{\theta}d\dot{\theta} = \frac{2K}{m}\sin\theta\, d\theta \tag{3.450}$$

Integrating Eq. (3.450) gives

$$\int_{\dot{\theta}_0}^{\dot{\theta}} \nu d\nu = \int_{\theta_0}^{\theta} \frac{2K}{m}\sin\eta d\eta \tag{3.451}$$

where ν and η are dummy variables of integration. We then obtain

$$\frac{\dot{\theta}^2 - \dot{\theta}_0^2}{2} = -\frac{2K}{m}(\cos\theta - \cos\theta_0) \tag{3.452}$$

Rearranging and simplifying Eq. (3.452) gives

$$\dot{\theta}^2 = \dot{\theta}_0^2 + \frac{4K}{m}(\cos\theta_0 - \cos\theta) \tag{3.453}$$

Substituting $\dot{\theta}^2$ from Eq. (3.453) into Eq. (3.448), we obtain

$$N = -mR\left[\dot{\theta}_0^2 + \frac{4K}{m}(\cos\theta_0 - \cos\theta)\right] + KR(1 + 2\cos\theta) \tag{3.454}$$

Simplifying Eq. (3.454) gives

$$N = -mR\dot{\theta}_0^2 - 4KR(\cos\theta_0 - \cos\theta) + KR(1 + 2\cos\theta) \tag{3.455}$$

Eq. (3.455) simplifies further to

$$N = -mR\dot{\theta}_0^2 - 4KR\cos\theta_0 + 6KR\cos\theta + KR \tag{3.456}$$

The reaction force exerted by the track on the particle is then given as

$$\mathbf{N} = \left[-mR\dot{\theta}_0^2 - 4KR\cos\theta_0 + 6KR\cos\theta + KR\right]\mathbf{e}_r \tag{3.457}$$

Question 3–20

A particle of mass m slides without friction along a fixed horizontal table as shown in Fig. P3-20. The particle is attached to an inextensible rope. The rope itself is threaded through a tiny hole in the table at point O such that the portion of the rope that hangs below the table remains vertical. Knowing that a constant vertical force $\mathbf{F}$ is applied to the rope, that the rope remains taut, and that gravity acts vertically downward, (a) determine a system of two differential equations in terms of r and θ describing the motion of the particle, (b) show that the angular momentum of the particle relative to point O is conserved, and (c) show that the total energy of the system is conserved.

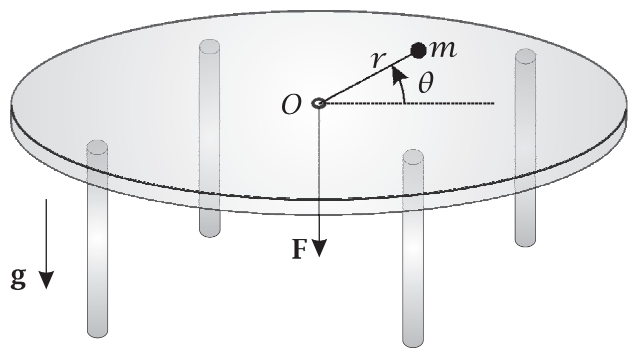

Figure P3-20

Solution to Question 3–20

Kinematics

First, let $\mathcal{F}$ be a reference frame fixed to the table. Then, choose the following coordinate system fixed in reference frame $\mathcal{F}$:

$$\begin{array}{ccr} & \text{Origin at } O & \\ \mathbf{E}_x & = & \text{Along } Om \text{ When } \theta = 0 \\ \mathbf{E}_z & = & \text{Orthogonal to Table} \\ \mathbf{E}_y & = & \mathbf{E}_z \times \mathbf{E}_x \end{array}$$

Next, let $\mathcal{A}$ be a reference frame fixed to the portion of the rope that lies on the table. Then, choose the following coordinate system fixed in reference frame $\mathcal{A}$:

$$\begin{array}{ccr} & \text{Origin at } O & \\ \mathbf{e}_r & = & \text{Along } Om \\ \mathbf{e}_z & = & \text{Orthogonal to Table} \\ \mathbf{e}_\theta & = & \mathbf{e}_z \times \mathbf{e}_r \end{array}$$

Then the position of the particle is given as

$$\mathbf{r} = r\mathbf{e}_r \tag{3.458}$$

Furthermore, the angular velocity of reference frame $\mathcal{A}$ in reference frame $\mathcal{F}$ is given as

$$^{\mathcal{F}}\boldsymbol{\omega}\mathcal{A} = \dot{\theta}\mathbf{e}_z \tag{3.459}$$

The velocity of the particle in reference frame $\mathcal{F}$ is then obtained from the rate of change transport theorem as

$$^{\mathcal{F}}\mathbf{v} = \frac{^{\mathcal{F}}d\mathbf{r}}{dt} = \frac{^{\mathcal{A}}d\mathbf{r}}{dt} + {}^{\mathcal{F}}\boldsymbol{\omega}^{\mathcal{A}} \times \mathbf{r} \tag{3.460}$$

Now we have that

$$\frac{^{\mathcal{A}}d\mathbf{r}}{dt} = \dot{r}\mathbf{e}_r \tag{3.461}$$

$$^{\mathcal{F}}\boldsymbol{\omega}^{\mathcal{A}} \times \mathbf{r} = \dot{\theta}\mathbf{e}_z \times r\mathbf{e}_r = r\dot{\theta}\mathbf{e}_\theta \tag{3.462}$$

Adding Eq. (3.461) and Eq. (3.462), we obtain the velocity of the particle in reference frame $\mathcal{F}$ as

$$^{\mathcal{F}}\mathbf{v} = \dot{r}\mathbf{e}_r + r\dot{\theta}\mathbf{e}_\theta \tag{3.463}$$

The acceleration of the particle in reference frame $\mathcal{F}$ is then obtained by applying the rate of change transport theorem to $^{\mathcal{F}}\mathbf{v}$ as

$$^{\mathcal{F}}\mathbf{a} = \frac{^{\mathcal{F}}d}{dt}\left(^{\mathcal{F}}\mathbf{v}\right) = \frac{^{\mathcal{A}}d}{dt}\left(^{\mathcal{F}}\mathbf{v}\right) + {}^{\mathcal{F}}\boldsymbol{\omega}^{\mathcal{A}} \times {}^{\mathcal{F}}\mathbf{v} \tag{3.464}$$

Now we have that

$$\frac{^{\mathcal{A}}d}{dt}\left(^{\mathcal{F}}\mathbf{v}\right) = \ddot{r}\mathbf{e}_r + (\dot{r}\dot{\theta} + r\ddot{\theta})\mathbf{e}_\theta \tag{3.465}$$

$$^{\mathcal{F}}\boldsymbol{\omega}^{\mathcal{A}} \times {}^{\mathcal{F}}\mathbf{v} = \dot{\theta}\mathbf{e}_z \times (\dot{r}\mathbf{e}_r + r\dot{\theta}\mathbf{e}_\theta) = \dot{r}\dot{\theta}\mathbf{e}_\theta - r\dot{\theta}^2\mathbf{e}_r \tag{3.466}$$

Adding Eq. (3.465) and Eq. (3.466), we obtain

$$^{\mathcal{F}}\mathbf{a} = (\ddot{r} - r\dot{\theta}^2)\mathbf{e}_r + (2\dot{r}\dot{\theta} + r\ddot{\theta})\mathbf{e}_\theta \tag{3.467}$$

Kinetics

The free body diagram of the particle is shown in Fig. 3-17. From Fig. 3-17 it can be seen that the three forces acting on the particle are given as

$$\begin{aligned} \mathbf{N} &= \text{Force of Table on Particle} \\ m\mathbf{g} &= \text{Force of Gravity} \\ \mathbf{F} &= \text{Force Exerted by Rope on Particle} \end{aligned}$$

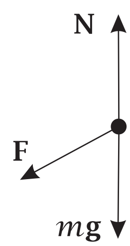

Figure 3-17 Free Body Diagram for Question 3–20.

Now we have that

$$\begin{aligned} \mathbf{N} &= N\mathbf{E}_z && (3.468) \\ m\mathbf{g} &= -mg\mathbf{E}_z && (3.469) \\ \mathbf{F} &= -F\mathbf{e}_r && (3.470) \end{aligned}$$

It is noted that, because the rope exerts a *known* force on the particle, it is necessary that the force $\mathbf{F}$ in Eq. (3.470) be in the *negative* $\mathbf{e}_r$-direction. Then, the resultant force acting on the particle (which we denote as $\mathbf{R}$ in order to avoid confusion with the given force $\mathbf{F}$) is given as

$$\mathbf{R} = \mathbf{N} + m\mathbf{g} + \mathbf{F} = N\mathbf{E}_z - mg\mathbf{E}_z + F\mathbf{e}_r = F\mathbf{e}_r + (N - mg)\mathbf{E}_z \quad (3.471)$$

Then, setting $\mathbf{R}$ in Eq. (3.471) equal to $m^{\mathcal{F}}\mathbf{a}$ using the expression for ${}^{\mathcal{F}}\mathbf{a}$ from Eq. (3.467), we obtain

$$-F\mathbf{e}_r + (N - mg)\mathbf{E}_z = m(\ddot{r} - r\dot{\theta}^2)\mathbf{e}_r + m(2\dot{r}\dot{\theta} + r\ddot{\theta})\mathbf{e}_\theta \quad (3.472)$$

Equating components in Eq. (3.472), we obtain the following three scalar equations:

$$\begin{aligned} -F &= m(\ddot{r} - r\dot{\theta}^2) && (3.473) \\ 0 &= m(2\dot{r}\dot{\theta} + r\ddot{\theta}) && (3.474) \\ 0 &= N - mg && (3.475) \end{aligned}$$

From Eq. (3.475) we have that

$$N = mg \quad (3.476)$$

Furthermore, since neither Eq. (3.473) nor Eq. (3.474) has any unknown reaction forces, these two equations are the differential equations of motion for the particle, i.e., a system of two differential equations of motion for the particle are given as

$$\begin{aligned} -F &= m(\ddot{r} - r\dot{\theta}^2) && (3.477) \\ 0 &= m(2\dot{r}\dot{\theta} + r\ddot{\theta}) && (3.478) \end{aligned}$$

Conservation of Angular Momentum Relative to Point O

First, it is important to observe that O is fixed in the inertial reference frame $\mathcal{F}$. Then, using the definition of the angular momentum of a particle relative to an inertially fixed point O, we have that

$$^{\mathcal{F}}\mathbf{H}_O = (\mathbf{r} - \mathbf{r}_0) \times m\,^{\mathcal{F}}\mathbf{v} \tag{3.479}$$

Then, noting that $\mathbf{r}_O = \mathbf{0}$ and substituting the expressions for $\mathbf{r}$ and $^{\mathcal{F}}\mathbf{v}$ from Eq. (3.458) and Eq. (3.463), respectively, into Eq. (3.479), we have that

$$^{\mathcal{F}}\mathbf{H}_O = r\mathbf{e}_r \times mr\dot{\theta}\mathbf{e}_\theta = mr^2\dot{\theta}\mathbf{E}_z \tag{3.480}$$

Now, in order to show that the angular momentum of the particle relative to point O will be conserved, we need to show that

$$\frac{^{\mathcal{F}}d}{dt}\left(^{\mathcal{F}}\mathbf{H}_O\right) = \mathbf{0} \tag{3.481}$$

Differentiating $^{\mathcal{F}}\mathbf{H}_O$ in Eq. (3.479) in reference frame $\mathcal{F}$, we have that

$$\frac{^{\mathcal{F}}d}{dt}\left(^{\mathcal{F}}\mathbf{H}_O\right) = m(2r\dot{r}\dot{\theta} + r^2\ddot{\theta})\mathbf{E}_z \tag{3.482}$$

where we note that $\mathbf{E}_z$ is a non-rotating direction. Then, using the second differential equation as given in Eq. (3.478) in Eq. (3.482), we obtain

$$\frac{^{\mathcal{F}}d}{dt}\left(^{\mathcal{F}}\mathbf{H}_O\right) = \mathbf{0} \tag{3.483}$$

which implies that $^{\mathcal{F}}\mathbf{H}_O$ is conserved.

Conservation of Energy

Applying the alternate form of the work-energy theorem for a particle in reference frame $\mathcal{F}$, we have that

$$\frac{d}{dt}\left(^{\mathcal{F}}E\right) = \mathbf{F}^{nc} \cdot {^{\mathcal{F}}\mathbf{v}} \tag{3.484}$$

Now, examining the free body diagram of the particle as given in Fig. 3-17, we see that the forces $m\mathbf{g}$ and $\mathbf{F}$ are conservative (since both of these forces are *constant*). Therefore, the only possible non-conservative force is $\mathbf{N}$. Consequently, we have that

$$\mathbf{F}^{nc} \cdot {^{\mathcal{F}}\mathbf{v}} = \mathbf{N} \cdot {^{\mathcal{F}}\mathbf{v}} \tag{3.485}$$

Substituting the expression for $\mathbf{N}$ from Eq. (3.468) and Eq. (3.463), we have that

$$\mathbf{F}^{nc} \cdot {^{\mathcal{F}}\mathbf{v}} = N\mathbf{E}_z \cdot (\dot{r}\mathbf{e}_r + r\dot{\theta}\mathbf{e}_\theta) = 0 \tag{3.486}$$

Substituting the result of Eq. (3.486) into Eq. (3.484), we have that

$$\frac{d}{dt}\left(^{\mathcal{F}}E\right) = 0 \tag{3.487}$$

which implies that energy is conserved.

Question 3–22

A particle of mass m is attached to a linear spring with spring constant K and unstretched length r_0 as shown in Fig. P3-22. The spring is attached at its other end to a massless collar where the collar slides along a frictionless horizontal track with a *known* displacement $x(t)$. Knowing that gravity acts downward, determine a system of two differential equations in terms of the variables r and θ that describe the motion of the particle.

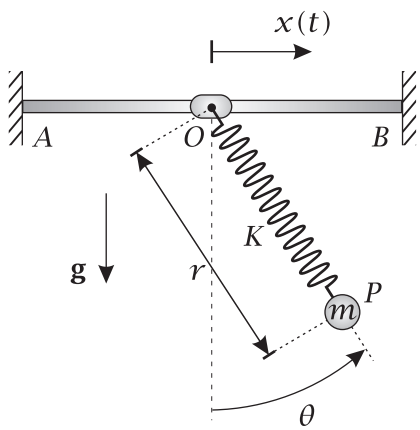

Figure P3-22

Solution to Question 3–22

Kinematics

First, let $\mathcal{F}$ be a reference frame fixed to the track. Then, choose the following coordinate system fixed in reference frame $\mathcal{F}$:

	Origin at Q When $x = 0$	
$\mathbf{E}_x$	=	To The Right
$\mathbf{E}_z$	=	Out of Page
$\mathbf{E}_y$	=	$\mathbf{E}_z \times \mathbf{E}_x$

Next, let $\mathcal{A}$ be a reference frame fixed to the direction of QP such that Q is a point fixed in reference frame $\mathcal{A}$. Then, choose the following coordinate system fixed in reference frame $\mathcal{A}$:

	Origin at O	
$\mathbf{e}_r$	=	Along QP
$\mathbf{e}_z$	=	Out of Page
$\mathbf{e}_\theta$	=	$\mathbf{E}_z \times \mathbf{e}_r$

The geometry of the bases $\{\mathbf{E}_x, \mathbf{E}_y, \mathbf{E}_z\}$ and $\{\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_z\}$ is shown in Fig. 3-18. Using Fig. 3-18, we have that

$$\begin{aligned} \mathbf{E}_x &= \sin\theta\,\mathbf{e}_r + \cos\theta\,\mathbf{e}_\theta & (3.488) \\ \mathbf{E}_y &= -\cos\theta\,\mathbf{e}_r + \sin\theta\,\mathbf{e}_\theta & (3.489) \end{aligned}$$

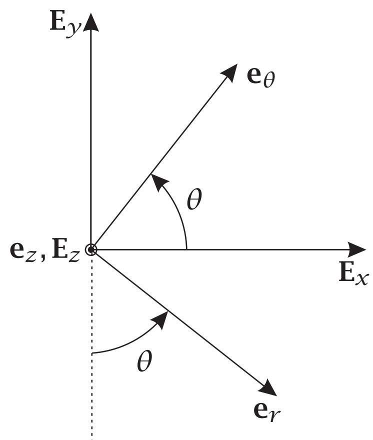

Figure 3-18 Geometry of Bases $\{\mathbf{E}_x, \mathbf{E}_y, \mathbf{E}_z\}$ and $\{\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_z\}$ for Question 3–22.

The position of the particle is then given as

$$\mathbf{r} = \mathbf{r}_Q + \mathbf{r}_{P/Q} \tag{3.490}$$

Now we have that

$$\begin{aligned} \mathbf{r}_Q &= x\mathbf{E}_x & (3.491) \\ \mathbf{r}_{P/Q} &= r\mathbf{e}_r & (3.492) \end{aligned}$$

Consequently, we obtain

$$\mathbf{r} = x\mathbf{E}_x + r\mathbf{e}_r \tag{3.493}$$

Next, the angular velocity of reference frame $\mathcal{A}$ in reference frame $\mathcal{F}$ is given as

$${}^{\mathcal{F}}\boldsymbol{\omega}^{\mathcal{A}} = \dot{\theta}\mathbf{e}_z \tag{3.494}$$

Then, the velocity of the particle in reference frame is obtained as

$${}^{\mathcal{F}}\mathbf{v} = \frac{{}^{\mathcal{F}}d\mathbf{r}}{dt} = \frac{{}^{\mathcal{F}}d}{dt}\left(\mathbf{r}_Q\right) + \frac{{}^{\mathcal{F}}d}{dt}\left(\mathbf{r}_{P/Q}\right) = {}^{\mathcal{F}}\mathbf{v}_Q + {}^{\mathcal{F}}\mathbf{v}_{P/Q} \tag{3.495}$$

Now since $\mathbf{r}_Q$ is expressed in terms of the basis $\{\mathbf{E}_x, \mathbf{E}_y, \mathbf{E}_z\}$ and $\{\mathbf{E}_x, \mathbf{E}_y, \mathbf{E}_z\}$ is fixed in reference frame $\mathcal{F}$, we have that

$${}^{\mathcal{F}}\mathbf{v}_Q = \dot{x}\mathbf{E}_x \tag{3.496}$$

Furthermore, since $\mathbf{r}_{P/Q}$ is expressed in terms of the basis $\{\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_z\}$ and $\{\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_z\}$ is fixed in reference frame $\mathcal{A}$, we can obtain ${}^{\mathcal{F}}\mathbf{v}_{P/Q}$ using the rate of change transport theorem between reference frame $\mathcal{A}$ and reference frame $\mathcal{F}$ as

$$ {}^{\mathcal{F}}\mathbf{v}_{P/Q} = \frac{{}^{\mathcal{F}}d}{dt}\left(\mathbf{r}_{P/Q}\right) = \frac{{}^{\mathcal{A}}d}{dt}\left(\mathbf{r}_{P/Q}\right) + {}^{\mathcal{F}}\boldsymbol{\omega}^{\mathcal{A}} \times \mathbf{r}_{P/Q} \tag{3.497} $$

Now we have that

$$ \frac{{}^{\mathcal{A}}d}{dt}\left(\mathbf{r}_{P/Q}\right) = \dot{r}\mathbf{e}_r \tag{3.498} $$

$$ {}^{\mathcal{F}}\boldsymbol{\omega}^{\mathcal{A}} \times \mathbf{r}_{P/Q} = \dot{\theta}\mathbf{e}_z \times r\mathbf{e}_r = r\dot{\theta}\mathbf{e}_\theta \tag{3.499} $$

Adding Eq. (3.498) and Eq. (3.499), we obtain ${}^{\mathcal{F}}\mathbf{v}_{P/Q}$ as

$$ {}^{\mathcal{F}}\mathbf{v}_{P/Q} = \dot{r}\mathbf{e}_r + r\dot{\theta}\mathbf{e}_\theta \tag{3.500} $$

Then, adding Eq. (3.496) and Eq. (3.500), the velocity of the particle in reference frame $\mathcal{F}$ is obtained as

$$ {}^{\mathcal{F}}\mathbf{v} = \dot{x}\mathbf{E}_x + \dot{r}\mathbf{e}_r + r\dot{\theta}\mathbf{e}_\theta \tag{3.501} $$

The acceleration of the particle in reference frame $\mathcal{F}$ is then obtained as

$$ {}^{\mathcal{F}}\mathbf{a} = \frac{{}^{\mathcal{F}}d}{dt}\left({}^{\mathcal{F}}\mathbf{v}\right) = \frac{{}^{\mathcal{F}}d}{dt}\left({}^{\mathcal{F}}\mathbf{v}_Q\right) + \frac{{}^{\mathcal{F}}d}{dt}\left({}^{\mathcal{F}}\mathbf{v}_{P/Q}\right) = {}^{\mathcal{F}}\mathbf{a}_Q + {}^{\mathcal{F}}\mathbf{a}_{P/Q} \tag{3.502} $$

Now we have that

$$ {}^{\mathcal{F}}\mathbf{a}_Q = \ddot{x}\mathbf{E}_x \tag{3.503} $$

Furthermore, since ${}^{\mathcal{F}}\mathbf{v}_{P/Q}$ is expressed in the basis $\{\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_z\}$ and $\{\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_z\}$ is fixed in reference frame $\mathcal{A}$, we obtain ${}^{\mathcal{F}}\mathbf{a}_{P/Q}$ using the rate of change transport theorem between reference frame $\mathcal{A}$ and reference frame $\mathcal{F}$ as

$$ {}^{\mathcal{F}}\mathbf{a}_{P/Q} = \frac{{}^{\mathcal{F}}d}{dt}\left({}^{\mathcal{F}}\mathbf{v}_{P/Q}\right) = \frac{{}^{\mathcal{A}}d}{dt}\left({}^{\mathcal{F}}\mathbf{v}_{P/Q}\right) + {}^{\mathcal{F}}\boldsymbol{\omega}^{\mathcal{A}} \times {}^{\mathcal{F}}\mathbf{v}_{P/Q} \tag{3.504} $$

Now we have that

$$ \frac{{}^{\mathcal{A}}d}{dt}\left({}^{\mathcal{F}}\mathbf{v}_{P/Q}\right) = \ddot{r}\mathbf{e}_r + (\dot{r}\dot{\theta} + r\ddot{\theta})\mathbf{e}_\theta \tag{3.505} $$

$$ {}^{\mathcal{F}}\boldsymbol{\omega}^{\mathcal{A}} \times {}^{\mathcal{F}}\mathbf{v}_{P/Q} = \dot{\theta}\mathbf{e}_z \times (\dot{r}\mathbf{e}_r + r\dot{\theta}\mathbf{e}_\theta) = -r\dot{\theta}^2\mathbf{e}_r + \dot{r}\dot{\theta}\mathbf{e}_\theta \tag{3.506} $$

Adding Eq. (3.505) and Eq. (3.506), we obtain

$$ {}^{\mathcal{F}}\mathbf{a}_{P/Q} = (\ddot{r} - r\dot{\theta}^2)\mathbf{e}_r + (2\dot{r}\dot{\theta} + r\ddot{\theta})\mathbf{e}_\theta \tag{3.507} $$

Then, adding Eq. (3.503) and Eq. (3.507), we obtain the acceleration of the particle in reference frame $\mathcal{F}$ as

$$ {}^{\mathcal{F}}\mathbf{a} = \ddot{x}\mathbf{E}_x + (\ddot{r} - r\dot{\theta}^2)\mathbf{e}_r + (2\dot{r}\dot{\theta} + r\ddot{\theta})\mathbf{e}_\theta \tag{3.508} $$

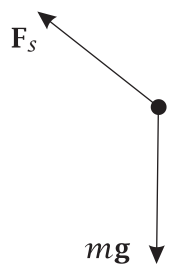

Figure 3-19 Free Body Diagram of Particle for Question 3–22.

Kinetics

The free body diagram of the particle is shown in Fig. 3-19.
Using Fig. 3-19. we see that the following forces act on the particle:

$$\begin{aligned} \mathbf{F}_s &= \text{Force of Spring} \\ m\mathbf{g} &= \text{Force of Gravity} \end{aligned}$$

First, we know that the model for a linear spring is

$$\mathbf{F}_s = -K(\ell - \ell_0)\mathbf{u}_s \tag{3.509}$$

where $\ell = \|\mathbf{r} - \mathbf{r}_Q\|$ and $\mathbf{u}_s = (\mathbf{r} - \mathbf{r}_Q)/\|\mathbf{r} - \mathbf{r}_Q\|$. Now for this problem we have that

$$\mathbf{r} - \mathbf{r}_Q = r\mathbf{e}_r \tag{3.510}$$

which implies that

$$\|\mathbf{r} - \mathbf{r}_Q\| = r \tag{3.511}$$

Therefore, we obtain

$$\mathbf{u}_s = \frac{r\mathbf{e}_r}{r} = \mathbf{e}_r \tag{3.512}$$

Furthermore, the unstretched length of the spring is $\ell_0 = r_0$. Consequently, the spring force is obtained as

$$\mathbf{F}_s = -K(r - r_0)\mathbf{e}_r \tag{3.513}$$

Next, the force of gravity is given as

$$m\mathbf{g} = -mg\mathbf{E}_y \tag{3.514}$$

Using the expression for $\mathbf{E}_y$ in terms of the basis $\{\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_z\}$ from Eq. (3.489), we obtain the force of gravity as

$$m\mathbf{g} = -mg(-\cos\theta\,\mathbf{e}_r + \sin\theta\,\mathbf{e}_\theta) = mg\cos\theta\,\mathbf{e}_r - mg\sin\theta\,\mathbf{e}_\theta \tag{3.515}$$

Then, adding Eq. (3.513) and Eq. (3.515), the resultant force acting on the particle is given as

$$\mathbf{F} = \mathbf{F}_s + m\mathbf{g} = -K(r - r_0)\mathbf{e}_r + mg\cos\theta\,\mathbf{e}_r - mg\sin\theta\,\mathbf{e}_\theta \tag{3.516}$$

Simplifying Eq. (3.516), we obtain

$$\mathbf{F} = \left[mg\sin\theta - K(r - r_0)\right]\mathbf{e}_r - mg\sin\theta\,\mathbf{e}_\theta \tag{3.517}$$

Then, setting $\mathbf{F}$ in Eq. (3.517) equal to $m^{\mathcal{F}}\mathbf{a}$ using the expression for ${}^{\mathcal{F}}\mathbf{a}$ from Eq. (3.508), we obtain

$$\left[mg\sin\theta - K(r - r_0)\right]\mathbf{e}_r - mg\sin\theta\,\mathbf{e}_\theta = m\left[\ddot{x}\mathbf{E}_x + (\ddot{r} - r\dot{\theta}^2)\mathbf{e}_r + (2\dot{r}\dot{\theta} + r\ddot{\theta})\mathbf{e}_\theta\right] \tag{3.518}$$

Then, substituting the expression for $\mathbf{E}_x$ from Eq. (3.488) into Eq. (3.518), we obtain

$$\begin{aligned}\left[mg\sin\theta - K(r - r_0)\right]\mathbf{e}_r - mg\sin\theta\,\mathbf{e}_\theta = m\,[&\ddot{x}(\sin\theta\,\mathbf{e}_r + \cos\theta\,\mathbf{e}_\theta) \\ &+ (\ddot{r} - r\dot{\theta}^2)\mathbf{e}_r + (2\dot{r}\dot{\theta} + r\ddot{\theta})\mathbf{e}_\theta\big]\end{aligned} \tag{3.519}$$

Rearranging Eq. (3.519) gives

$$\begin{aligned}\left[mg\sin\theta - K(r - r_0)\right]\mathbf{e}_r - mg\sin\theta\,\mathbf{e}_\theta = {}& m(\ddot{x}\sin\theta + \ddot{r} - r\dot{\theta}^2)\mathbf{e}_r \\ &+ m(\ddot{x}\cos\theta + 2\dot{r}\dot{\theta} + r\ddot{\theta})\mathbf{e}_\theta\end{aligned} \tag{3.520}$$

Equating components in Eq. (3.520), we obtain

$$m\ddot{x}\sin\theta + m\ddot{r} - mr\dot{\theta}^2 = mg\sin\theta - K(r - r_0) \tag{3.521}$$

$$m\ddot{x}\cos\theta + 2m\dot{r}\dot{\theta} + mr\ddot{\theta} = -mg\sin\theta \tag{3.522}$$

Simplifying and rearranging Eq. (3.521) and Eq. (3.522), we obtain a system of two differential equations describing the motion of the particle as

$$\ddot{r} - r\dot{\theta}^2 - g\sin\theta + \frac{K}{m}(r - r_0) = -\ddot{x}\sin\theta \tag{3.523}$$

$$r\ddot{\theta} + 2\dot{r}\dot{\theta} + g\sin\theta = -\ddot{x}\cos\theta \tag{3.524}$$

Question 3–23

A collar of mass m slides with friction along a rod that is welded rigidly at a constant angle β with the vertical to a shaft as shown in Fig. P3-23. The shaft rotates about the vertical with constant angular velocity $\mathbf{\Omega}$ (where $\Omega = \|\mathbf{\Omega}\|$). Knowing that r is the radial distance from point of the weld to the collar, that the friction is viscous with viscous friction coefficient c, and that gravity acts vertically downward, determine the differential equation of motion for the collar in terms of r.

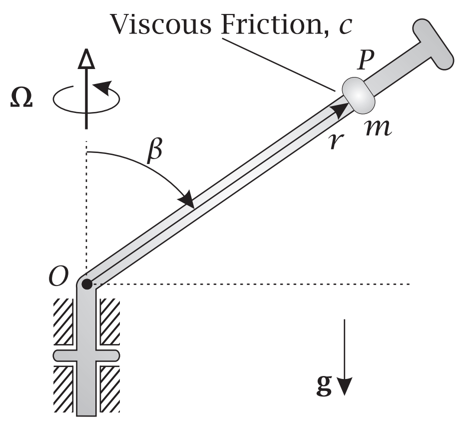

Figure P3-23

Solution to Question 3–23

Kinematics

First, let $\mathcal{F}$ be a fixed reference frame. Then, choose the following coordinate system fixed in reference frame $\mathcal{F}$:

Origin at O		
$\mathbf{E}_x$	=	To the Right at$t = 0$
$\mathbf{E}_z$	=	Into Page at $t = 0$
$\mathbf{E}_y$	=	$\mathbf{E}_z \times \mathbf{E}_x$ = Up

Next, let $\mathcal{A}$ be a reference frame fixed to the shaft and tube. Then, choose the following coordinate system fixed in reference frame $\mathcal{A}$:

Origin at O		
$\mathbf{e}_r$	=	along OP
$\mathbf{e}_z$	=	Orthogonal to Plane of Shaft and Arm and Into Page
$\mathbf{e}_\theta$	=	$\mathbf{e}_z \times \mathbf{e}_r$

The geometry of the bases $\{\mathbf{E}_x, \mathbf{E}_y, \mathbf{E}_z\}$ and $\{\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_z\}$ is shown in Fig. 3-20. Using Fig. 3-20, the relationship between the basis $\{\mathbf{E}_x, \mathbf{E}_y, \mathbf{E}_z\}$ and $\{\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_z\}$

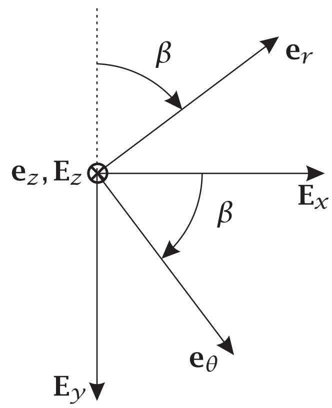

Figure 3-20 Geometry for Question 3–23.

is

$$\mathbf{E}_x = \sin\beta\mathbf{e}_r + \cos\beta\mathbf{e}_\theta \quad (3.525)$$

$$\mathbf{E}_y = -\cos\beta\mathbf{e}_r + \sin\beta\mathbf{e}_\theta \quad (3.526)$$

The position of the particle is then given in terms of the basis $\{\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_z\}$ as

$$\mathbf{r} = r\mathbf{e}_r \quad (3.527)$$

Furthermore, the angular velocity of reference frame $\mathcal{A}$ in reference frame $\mathcal{F}$ is then given as

$$^{\mathcal{F}}\boldsymbol{\omega}^{\mathcal{A}} = \boldsymbol{\Omega} = -\Omega\mathbf{E}_y \quad (3.528)$$

Then, substituting the expression for $\mathbf{E}_y$ from Eq. (3.526) into Eq. (3.528), we obtain

$$^{\mathcal{F}}\boldsymbol{\omega}^{\mathcal{A}} = -\Omega(-\cos\beta\mathbf{e}_r + \sin\beta\mathbf{e}_\theta) = \Omega\cos\beta\mathbf{e}_r - \Omega\sin\beta\mathbf{e}_\theta \quad (3.529)$$

The velocity of the particle in reference frame $\mathcal{F}$ is then obtained from the transport theorem as

$$^{\mathcal{F}}\mathbf{v} = \frac{^{\mathcal{F}}d\mathbf{r}}{dt} = \frac{^{\mathcal{A}}d\mathbf{r}}{dt} + {}^{\mathcal{F}}\boldsymbol{\omega}^{\mathcal{A}} \times \mathbf{r} \quad (3.530)$$

Now we have that

$$\frac{^{\mathcal{A}}d\mathbf{r}}{dt} = \dot{r}\mathbf{e}_r \quad (3.531)$$

$$^{\mathcal{F}}\boldsymbol{\omega}^{\mathcal{A}} \times \mathbf{r} = (\Omega\cos\beta\mathbf{e}_r - \Omega\sin\beta\mathbf{e}_\theta) \times r\mathbf{e}_r = r\Omega\sin\beta\mathbf{e}_z \quad (3.532)$$

Adding the expressions in Eq. (3.531) and Eq. (3.532), we obtain the velocity of the collar in reference frame $\mathcal{F}$ as

$$^{\mathcal{F}}\mathbf{v} = \dot{r}\mathbf{e}_r + r\Omega\sin\beta\mathbf{e}_z \tag{3.533}$$

The acceleration of the collar in reference frame $\mathcal{F}$ is obtained by applying the rate of change transport theorem as

$$^{\mathcal{F}}\mathbf{a} = \frac{^{\mathcal{F}}d}{dt}\left(^{\mathcal{F}}\mathbf{v}\right) = \frac{^{\mathcal{A}}d}{dt}\left(^{\mathcal{F}}\mathbf{v}\right) + {}^{\mathcal{F}}\boldsymbol{\omega}^{\mathcal{A}} \times {}^{\mathcal{F}}\mathbf{v} \tag{3.534}$$

Now we have that

$$\frac{^{\mathcal{A}}d}{dt}\left(^{\mathcal{F}}\mathbf{v}\right) = \ddot{r}\mathbf{e}_r + \dot{r}\Omega\sin\beta\mathbf{e}_z \tag{3.535}$$

$$\begin{aligned} {}^{\mathcal{F}}\boldsymbol{\omega}^{\mathcal{A}} \times {}^{\mathcal{F}}\mathbf{v} &= (\Omega\cos\beta\mathbf{e}_r - \Omega\sin\beta\mathbf{e}_\theta) \times (\dot{r}\mathbf{e}_r + r\Omega\sin\beta\mathbf{e}_z) \\ &= -r\Omega^2\cos\beta\sin\beta\mathbf{e}_\theta + \dot{r}\Omega\sin\beta\mathbf{e}_z - r\Omega^2\sin^2\beta\mathbf{e}_r \end{aligned} \tag{3.536}$$

Adding the expressions in Eq. (3.535) and Eq. (3.536), we obtain the acceleration of the collar in reference frame $\mathcal{F}$ as

$$^{\mathcal{F}}\mathbf{a} = \ddot{r}\mathbf{e}_r + \dot{r}\Omega\sin\beta\mathbf{e}_z - r\Omega^2\cos\beta\sin\beta\mathbf{e}_\theta + \dot{r}\Omega\sin\beta\mathbf{e}_z - r\Omega^2\sin^2\beta\mathbf{e}_r \tag{3.537}$$

Simplifying Eq. (3.537) gives

$$^{\mathcal{F}}\mathbf{a} = (\ddot{r} - r\Omega^2\sin^2\beta)\mathbf{e}_r - r\Omega^2\cos\beta\sin\beta\mathbf{e}_\theta + 2\dot{r}\Omega\sin\beta\mathbf{e}_z \tag{3.538}$$

Kinetics

The free body diagram of the particle is shown in Fig. 3-21 Using Fig. 3-21, we

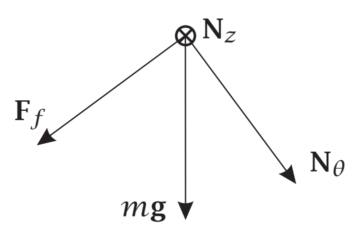

Figure 3-21 Free Body Diagram for Question 3–23.

have that

$$\begin{aligned} \mathbf{N}_\theta &= \text{Reaction Force of Rod on Particle in } \mathbf{e}_\theta - \text{Direction} \\ \mathbf{N}_z &= \text{Reaction Force of Rod on Particle in } \mathbf{e}_z - \text{Direction} \\ m\mathbf{g} &= \text{Force of Gravity} \\ \mathbf{F}_f &= \text{Force of Viscous Friction} \end{aligned}$$

Given the geometry of the problem, we have that

$$\begin{aligned} \mathbf{N}_\theta &= N_\theta \mathbf{e}_\theta \qquad (3.539)\\ \mathbf{N}_z &= N_z \mathbf{e}_z \qquad (3.540)\\ m\mathbf{g} &= -mg\mathbf{E}_y \qquad (3.541)\\ \mathbf{F}_f &= -c\mathbf{v}_{\text{rel}} \qquad (3.542)\end{aligned}$$

where $\mathbf{v}_{\text{rel}}$ is given as

$$\mathbf{v}_{\text{rel}} = {}^{\mathcal{F}}\mathbf{v} - {}^{\mathcal{F}}\mathbf{v}_P^{\mathcal{A}} \qquad (3.543)$$

where ${}^{\mathcal{F}}\mathbf{v}_P^{\mathcal{A}}$ is the velocity of the point P on the rod that is instantaneously in contact with the collar. Now we have that

$${}^{\mathcal{F}}\mathbf{v}_P^{\mathcal{A}} = {}^{\mathcal{A}}\mathbf{v}_P^{\mathcal{A}} + {}^{\mathcal{F}}\boldsymbol{\omega}^{\mathcal{A}} \times \mathbf{r} \qquad (3.544)$$

However, since ${}^{\mathcal{F}}\mathbf{v}_P^{\mathcal{A}}$ corresponds to the velocity of a point fixed to the rod, we have that

$${}^{\mathcal{A}}\mathbf{v}_P^{\mathcal{A}} = \mathbf{0} \qquad (3.545)$$

Therefore,

$${}^{\mathcal{F}}\mathbf{v}_P^{\mathcal{A}} = {}^{\mathcal{F}}\boldsymbol{\omega}^{\mathcal{A}} \times \mathbf{r} = (\Omega\cos\beta\mathbf{e}_r - \Omega\sin\beta\mathbf{e}_\theta) \times r\mathbf{e}_r = r\Omega\sin\beta\mathbf{e}_z \qquad (3.546)$$

Then, substituting the result of Eq. (3.546) and the expression for ${}^{\mathcal{F}}\mathbf{v}$ from Eq. (3.533) into Eq. (3.543), we obtain

$$\mathbf{v}_{\text{rel}} = \dot{r}\mathbf{e}_r + r\Omega\sin\beta\mathbf{e}_z - r\Omega\sin\beta\mathbf{e}_z = \dot{r}\mathbf{e}_r \qquad (3.547)$$

The force of viscous friction acting on the collar is then given as

$$\mathbf{F}_f = -c\mathbf{v}_{\text{rel}} = -c\dot{r}\mathbf{e}_r \qquad (3.548)$$

The resultant force acting on the particle is then given as

$$\mathbf{F} = \mathbf{N}_\theta + \mathbf{N}_z + m\mathbf{g} + \mathbf{F}_f = N_\theta\mathbf{e}_\theta + N_z\mathbf{e}_z + mg\mathbf{E}_y - c\dot{r}\mathbf{e}_r \qquad (3.549)$$

Then, substituting the expression for $\mathbf{E}_y$ from Eq. (3.526) into Eq. (3.549), we obtain

$$\begin{aligned} \mathbf{F} &= N_\theta\mathbf{e}_\theta + N_z\mathbf{e}_z + mg(-\cos\beta\mathbf{e}_r + \sin\beta\mathbf{e}_\theta) - c\dot{r}\mathbf{e}_r \\ &= -(mg\cos\beta + c\dot{r})\mathbf{e}_r + (N_\theta + mg\sin\beta)\mathbf{e}_\theta + N_z\mathbf{e}_z \end{aligned} \qquad (3.550)$$

Then, applying Newton's 2^{nd} Law $\mathbf{F}$ from Eq. (3.550) and ${}^{\mathcal{F}}\mathbf{a}$ from Eq. (3.538), we obtain

$$\begin{aligned} -(mg\cos\beta + c\dot{r})\mathbf{e}_r + (N_\theta + mg\sin\beta)\mathbf{e}_\theta + N_z\mathbf{e}_z = {} & m(\ddot{r} - r\Omega^2\sin^2\beta)\mathbf{e}_r \\ & - mr\Omega^2\cos\beta\sin\beta\mathbf{e}_\theta \\ & + 2m\dot{r}\Omega\sin\beta\mathbf{e}_z \end{aligned} \qquad (3.551)$$

Equating components in Eq. (3.551), we obtain the following three scalar equations:

$$
\begin{aligned}
m(\ddot{r} - r\Omega^2 \sin^2\beta) &= -(mg\cos\beta + c\dot{r}) && (3.552)\\
-mr\Omega^2 \cos\beta \sin\beta &= N_\theta - mg\sin\theta && (3.553)\\
2m\dot{r}\Omega\sin\beta &= N_z && (3.554)
\end{aligned}
$$

It can be seen that Eq. (3.552) has no reaction forces. Furthermore, in this problem only one variable is required to describe the motion. Consequently, the differential equation of motion is given as

$$
m(\ddot{r} - r\Omega^2 \sin^2\beta) = -(mg\cos\beta + c\dot{r}) \tag{3.555}
$$

Rearranging and simplifying Eq. (3.555), we obtain the differential equation of motion as

$$
\ddot{r} - r\Omega^2 \sin^2\beta + c\dot{r} + g\cos\beta = 0 \tag{3.556}
$$

Question 3–25

A particle of mass m slides without friction along a track in the form of a parabola as shown in Fig. P3-25. The equation for the parabola is

$$y = \frac{r^2}{2a}$$

where a is a constant, r is the distance from point O to point Q, point Q is the projection of point P onto the horizontal direction, and y is the vertical distance. Furthermore, the particle is attached to a linear spring with spring constant K and unstretched length x_0. The spring is always aligned horizontally such that its attachment point is free to slide along a vertical shaft through the center of the parabola. Knowing that the parabola rotates with constant angular velocity $\mathbf{\Omega}$ (where $\Omega = \|\mathbf{\Omega}\|$) about the vertical direction and that gravity acts vertically downward, determine the differential equation of motion for the particle in terms of the variable r.

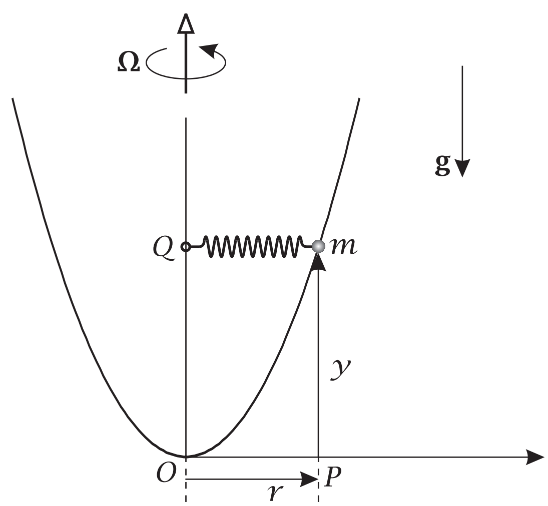

Figure P3-25

Solution to Question 3–25

Kinematics

For this problem it is convenient to define a fixed inertial reference frame $\mathcal{F}$ and a non-inertial reference frame $\mathcal{A}$. Corresponding to reference frame $\mathcal{F}$, we choose the following coordinate system:

Origin at Point O

$$\begin{array}{lcl} \mathbf{E}_x & = & \text{Along } OP \text{ When } t = 0 \\ \mathbf{E}_y & = & \text{Along } Oy \text{ When } t = 0 \\ \mathbf{E}_z & = & \mathbf{E}_x \times \mathbf{E}_y \end{array}$$

Furthermore, corresponding to reference frame $\mathcal{A}$, we choose the following coordinate system:

Origin at Point O

$$\begin{array}{lcl} \mathbf{e}_x & = & \text{Along } OP \\ \mathbf{e}_y & = & \text{Along } Oy \\ \mathbf{e}_z & = & \mathbf{e}_x \times \mathbf{e}_y \end{array}$$

The position of the particle is then given in terms of the basis $\{\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z\}$ as

$$\mathbf{r} = x\mathbf{e}_x + y\mathbf{e}_y = x\mathbf{e}_x + (x^2/a)\mathbf{e}_y \tag{3.557}$$

Furthermore, since the parabola spins about the ey-direction, the angular velocity of reference frame $\mathcal{A}$ in reference frame $\mathcal{F}$ is given as

$$^{\mathcal{F}}\boldsymbol{\omega}^{\mathcal{A}} = \boldsymbol{\Omega} = \Omega\mathbf{e}_y \tag{3.558}$$

The velocity in reference frame $\mathcal{F}$ is then found using the rate of change transport theorem as

$$^{\mathcal{F}}\mathbf{v} = \frac{^{\mathcal{F}}d\mathbf{r}}{dt} = \frac{^{\mathcal{A}}d\mathbf{r}}{dt} + {}^{\mathcal{F}}\boldsymbol{\omega}^{\mathcal{A}} \times \mathbf{r} \tag{3.559}$$

Using $\mathbf{r}$ from Eq. (3.557) and $^{\mathcal{F}}\boldsymbol{\omega}^{\mathcal{A}}$ from Eq. (3.558), we have that

$$\frac{^{\mathcal{A}}d\mathbf{r}}{dt} = \dot{x}\mathbf{e}_x + (2x\dot{x}/a)\mathbf{e}_y \tag{3.560}$$

$$^{\mathcal{F}}\boldsymbol{\omega}^{\mathcal{A}} \times \mathbf{r} = \Omega\mathbf{e}_y \times (x\mathbf{e}_x + (x^2/a)\mathbf{e}_y) = -\Omega x\mathbf{e}_z \tag{3.561}$$

Adding the expressions in Eq. (3.560) and Eq. (3.561), we obtain $^{\mathcal{F}}\mathbf{v}$ as

$$^{\mathcal{F}}\mathbf{v} = \dot{x}\mathbf{e}_x + (2x\dot{x}/a)\mathbf{e}_y - \Omega x\mathbf{e}_z \tag{3.562}$$

The acceleration in reference frame $\mathcal{F}$ is found by applying the rate of change transport theorem to $^{\mathcal{F}}\mathbf{v}$ as

$$^{\mathcal{F}}\mathbf{a} = \frac{^{\mathcal{F}}d}{dt}\left({}^{\mathcal{F}}\mathbf{v}\right) = \frac{^{\mathcal{A}}d}{dt}\left({}^{\mathcal{F}}\mathbf{v}\right) + {}^{\mathcal{F}}\boldsymbol{\omega}^{\mathcal{A}} \times {}^{\mathcal{F}}\mathbf{v} \tag{3.563}$$

Using $^{\mathcal{F}}\mathbf{v}$ from Eq. (3.562) and $^{\mathcal{F}}\boldsymbol{\omega}^{\mathcal{A}}$ from Eq. (3.558), we have that

$$\frac{^{\mathcal{A}}d}{dt}\left({}^{\mathcal{F}}\mathbf{v}\right) = \ddot{x}\mathbf{e}_x + \left[2(\dot{x}^2 + x\ddot{x})/a\right]\mathbf{e}_y - \Omega\dot{x}\mathbf{e}_z \tag{3.564}$$

$$^{\mathcal{F}}\boldsymbol{\omega}^{\mathcal{A}} \times {}^{\mathcal{F}}\mathbf{v} = \Omega\mathbf{e}_y \times (\dot{x}\mathbf{e}_x + (2x\dot{x}/a)\mathbf{e}_y - \Omega x\mathbf{e}_z) = -\Omega\dot{x}\mathbf{e}_z - \Omega^2 x\mathbf{e}_x \tag{3.565}$$

Adding the expressions in Eq. (3.564) and Eq. (3.565) and, we obtain $^{\mathcal{F}}\mathbf{a}$ as

$$^{\mathcal{F}}\mathbf{a} = (\ddot{x} - \Omega^2 x)\mathbf{e}_x + \left[2(\dot{x}^2 + x\ddot{x})/a\right]\mathbf{e}_y - 2\Omega\dot{x}\mathbf{e}_z \tag{3.566}$$

Kinetics

The free body diagram of the particle is shown in Fig. 3-22. Using Fig. 3-22, it is

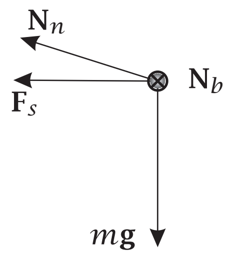

Figure 3-22 Free Body Diagram of Particle for Question 3–25.

seen that the following forces act on the particle:

$$\begin{array}{lcl} \mathbf{N}_n & = & \text{Reaction Force of Track on Particle} \\ & & \text{Normal to Track and In Plane of Parabola} \\ \mathbf{N}_b & = & \text{Reaction Force of Track on Particle} \\ & & \text{Normal to Track and Orthogonal to Plane of Parabola} \\ \mathbf{F}_s & = & \text{Spring Force} \\ m\mathbf{g} & = & \text{Force of Gravity} \end{array}$$

Given the description of the two reaction forces $\mathbf{N}_n$ and $\mathbf{N}_b$, we have that

$$\mathbf{N}_n = N_n \mathbf{e}_n \tag{3.567}$$
$$\mathbf{N}_b = N_b \mathbf{e}_b \tag{3.568}$$

where $\mathbf{e}_n$ and $\mathbf{e}_b$ are the principle unit normal and principle unit bi-normal vector *to the parabola*. Now since reference frame $\mathcal{A}$ is the reference frame of the parabola, the tangent vector to the parabola is given as

$$\mathbf{e}_t = \frac{{}^{\mathcal{A}}\mathbf{v}}{\|{}^{\mathcal{A}}\mathbf{v}\|} \tag{3.569}$$

Now we have ${}^{\mathcal{A}}\mathbf{v}$ from Eq. (3.560) as

$${}^{\mathcal{A}}\mathbf{v} = \dot{x}\mathbf{e}_x + (2x\dot{x}/a)\mathbf{e}_y \tag{3.570}$$

Consequently,

$$\mathbf{e}_t = \frac{\dot{x}\mathbf{e}_x + (2x\dot{x}/a)\mathbf{e}_y}{\dot{x}\sqrt{1 + \left(\dfrac{2x}{a}\right)^2}} = \frac{\mathbf{e}_x + (2x/a)\mathbf{e}_y}{\sqrt{1 + \left(\dfrac{2x}{a}\right)^2}} \tag{3.571}$$

Next, we know that $\mathbf{e}_b$ must lie orthogonal to the plane of the parabola. Consequently, we have that

$$\mathbf{e}_b = \mathbf{e}_z \tag{3.572}$$

Therefore,

$$\mathbf{e}_n = \mathbf{e}_b \times \mathbf{e}_t = \mathbf{e}_z \times \frac{\mathbf{e}_x + (2x/a)\mathbf{e}_y}{\sqrt{1 + \left(\frac{2x}{a}\right)^2}} = \frac{-(2x/a)\mathbf{e}_x + \mathbf{e}_y}{\sqrt{1 + \left(\frac{2x}{a}\right)^2}} \tag{3.573}$$

Suppose now that we define

$$\gamma = \sqrt{1 + \left(\frac{2x}{a}\right)^2} \tag{3.574}$$

Then we can write

$$\mathbf{e}_n = \frac{-(2x/a)\mathbf{e}_x + \mathbf{e}_y}{\gamma} \tag{3.575}$$

The reaction force exerted by the parabola on the particle is then given as

$$\mathbf{N} = \mathbf{N}_n + \mathbf{N}_b = N_n \frac{-(2x/a)\mathbf{e}_x + \mathbf{e}_y}{\gamma} + N_b \mathbf{e}_z \tag{3.576}$$

Next, the spring force is given as

$$\mathbf{F}_s = -K(\ell - \ell_0)\mathbf{u}_s \tag{3.577}$$

Now for this problem we know that the unstretched length of the spring is $\ell_0 = x_0$. Furthermore, the stretched length of the spring is given as

$$\ell = \|\mathbf{r} - \mathbf{r}_Q\| \tag{3.578}$$

where P is the attachment point of the spring. Now since the attachment point lies on the $\mathbf{e}_y$-axis at the same value of y as the particle, we have that

$$\mathbf{r}_Q = y\mathbf{e}_y = \frac{x^2}{a}\mathbf{e}_y \tag{3.579}$$

Therefore,

$$\mathbf{r} - \mathbf{r}_Q = x\mathbf{e}_x + \frac{x^2}{a}\mathbf{e}_y - \frac{x^2}{a}\mathbf{e}_y = x\mathbf{e}_x \tag{3.580}$$

The stretched length of the spring is then given as

$$\ell = \|x\mathbf{e}_x\| = x \tag{3.581}$$

Finally, the direction of the spring force is given as

$$\mathbf{u}_s = \frac{\mathbf{r} - \mathbf{r}_Q}{\|\mathbf{r} - \mathbf{r}_Q\|} = \frac{x\mathbf{e}_x}{x} = \mathbf{e}_x \tag{3.582}$$

The force of the linear spring is then given as

$$\mathbf{F}_s = -K(x - x_0)\mathbf{e}_x \tag{3.583}$$

Finally, the force of gravity is given as

$$m\mathbf{g} = -mg\mathbf{e}_y \tag{3.584}$$

Then, adding Eq. (3.576), Eq. (3.583), and Eq. (3.584), the resultant force acting on the particle is then obtained as

$$\mathbf{F} = \mathbf{N}_n + \mathbf{N}_b + \mathbf{F}_s + m\mathbf{g} = N_n \frac{-(2x/a)\mathbf{e}_x + \mathbf{e}_y}{\gamma} + N_b\mathbf{e}_z - K(x - x_0)\mathbf{e}_x - mg\mathbf{e}_y \tag{3.585}$$

Then, combining terms with common components in Eq. (3.585), we have that

$$\mathbf{F} = -\left[N_n \frac{2x/a}{\gamma} + K(x - x_0)\right]\mathbf{e}_x + \left[\frac{N_n}{\gamma} - mg\right]\mathbf{e}_y + N_b\mathbf{e}_z \tag{3.586}$$

Then, setting $\mathbf{F}$ from Eq. (3.586) equal to $m{}^{\mathcal{F}}\mathbf{a}$ using the expression for ${}^{\mathcal{F}}\mathbf{a}$ from Eq. (3.566), we obtain

$$\begin{aligned} -\left[N_n \frac{2x/a}{\gamma} + K(x - x_0)\right]\mathbf{e}_x + \left[\frac{N_n}{\gamma} - mg\right]\mathbf{e}_y + N_b\mathbf{e}_z = m(\ddot{x} - \Omega^2 x)\mathbf{e}_x \\ + m\left[2(\dot{x}^2 + x\ddot{x})/a\right]\mathbf{e}_y \\ - 2m\Omega\dot{x}\mathbf{e}_z \end{aligned} \tag{3.587}$$

Equating components in Eq. (3.587), we obtain the following three scalar equations:

$$m(\ddot{x} - \Omega^2 x) = -N_n \frac{2x/a}{\gamma} - K(x - x_0) \tag{3.588}$$

$$m\left[2(\dot{x}^2 + x\ddot{x})/a\right] = \frac{N_n}{\gamma} - mg \tag{3.589}$$

$$-2m\Omega\dot{x} = N_b \tag{3.590}$$

Now the differential equation of motion for the particle is obtained as follows. Rearranging Eq. (3.588) and Eq. (3.589), we have that

$$N_n \frac{2x/a}{\gamma} = -m(\ddot{x} - \Omega^2 x) - K(x - x_0) \tag{3.591}$$

$$\frac{N_n}{\gamma} = m\left[2(\dot{x}^2 + x\ddot{x})/a\right] + mg \tag{3.592}$$

Then, dividing Eq. (3.591) by Eq. (3.592), we obtain

$$\frac{N_n \dfrac{2x/a}{\gamma}}{\dfrac{N_n}{\gamma}} = \frac{-m(\ddot{x} - \Omega^2 x) - K(x - x_0)}{m\left[2(\dot{x}^2 + x\ddot{x})/a\right] + mg} \tag{3.593}$$

Simplifying Eq. (3.593) gives

$$\frac{2x}{a} = \frac{-m(\ddot{x} - \Omega^2 x) - K(x - x_0)}{m\left[2(\dot{x}^2 + x\ddot{x})/a\right] + mg} \tag{3.594}$$

Multiplying Eq. (3.594) by $m\left[2(\dot{x}^2 + x\ddot{x})/a\right] + mg$, we obtain

$$\left(m\left[2(\dot{x}^2 + x\ddot{x})/a\right] + mg\right)\frac{2x}{a} = -m(\ddot{x} - \Omega^2 x) - K(x - x_0) \tag{3.595}$$

Rearranging Eq. (3.595), we have that

$$m(\ddot{x} - \Omega^2 x) + K(x - x_0) + \left(m\left[2(\dot{x}^2 + x\ddot{x})/a\right] + mg\right)\frac{2x}{a} = 0 \tag{3.596}$$

Simplifying further, we obtain the differential equation of motion as

$$m\ddot{x}\left[1 + \left(\frac{2x}{a}\right)^2\right] + m\left[\frac{2g}{a} - \Omega^2\right]x + K(x - x_0) + 4mx\left(\frac{\dot{x}}{a}\right)^2 = 0 \tag{3.597}$$