Chapter 4

Kinetics of a System of Particles

Question 4–1

A particle of mass m is connected to a block of mass M via a rigid massless rod of length l as shown in Fig. P4-1. The rod is free to pivot about a hinge attached to the block at point O. Furthermore, the block rolls without friction along a horizontal surface. Knowing that a horizontal force **F** is applied to the block and that gravity acts downward, determine a system of two differential equations describing the motion of the block and the particle.

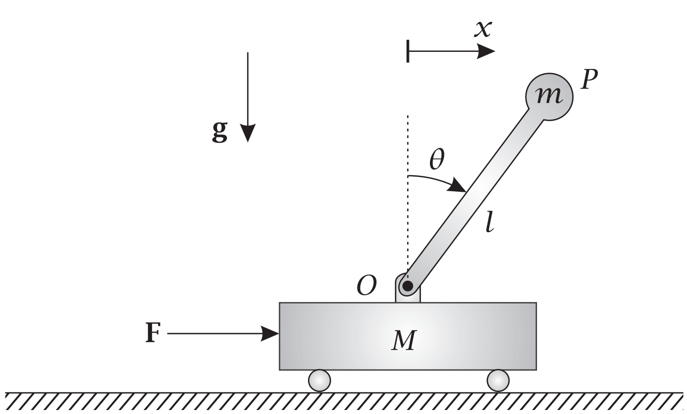

Figure P4-1

Solution to Question 4–1

Kinematics

Let $\mathcal{F}$ be a reference frame fixed to the ground. Then, choose the following coordinate system fixed in reference frame $\mathcal{F}$:

$$\begin{array}{ccc} & \text{Origin at } O \text{ at } t=0 & \\ \mathbf{E}_x & = & \text{To the Right} \\ \mathbf{E}_z & = & \text{Into Page} \\ \mathbf{E}_y & = & \mathbf{E}_z \times \mathbf{E}_x \end{array}$$

Next, let $\mathcal{A}$ be a reference frame fixed to the rod. Then, choose the following coordinate system fixed in reference frame $\mathcal{A}$:

$$\begin{array}{ccc} & \text{Origin at } O & \\ \mathbf{e}_r & = & \text{Along } OP \\ \mathbf{e}_z & = & \text{Into Page} \\ \mathbf{e}_\theta & = & \mathbf{E}_z \times \mathbf{e}_r \end{array}$$

We note that the relationship between the basis $\{\mathbf{E}_x, \mathbf{E}_y, \mathbf{E}_z\}$ and $\{\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_z\}$ is given as

$$\mathbf{E}_x = \sin\theta\,\mathbf{e}_r + \cos\theta\,\mathbf{e}_\theta \tag{4.1}$$

$$\mathbf{E}_y = -\cos\theta\,\mathbf{e}_r + \sin\theta\,\mathbf{e}_\theta \tag{4.2}$$

Also, we have that

$$\mathbf{e}_r = \sin\theta\,\mathbf{E}_x - \cos\theta\,\mathbf{E}_y \tag{4.3}$$

$$\mathbf{e}_\theta = \cos\theta\,\mathbf{E}_x + \sin\theta\,\mathbf{E}_y \tag{4.4}$$

Using the bases $\{\mathbf{E}_x, \mathbf{E}_y, \mathbf{E}_z\}$ and $\{\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_z\}$, the position of the block is given as

$$\mathbf{r}_O = x\mathbf{E}_x \tag{4.5}$$

Then the velocity and acceleration of the block in reference frame $\mathcal{F}$ are given, respectively, as

$${}^{\mathcal{F}}\mathbf{v}_O = \dot{x}\mathbf{E}_x \tag{4.6}$$

$${}^{\mathcal{F}}\mathbf{a}_O = \ddot{x}\mathbf{E}_x \tag{4.7}$$

Next, the position of the particle is given as

$$\mathbf{r} = \mathbf{r}_P = \mathbf{r}_O + \mathbf{r}_{P/O} = x\mathbf{E}_x + l\mathbf{e}_r \tag{4.8}$$

Next, the angular velocity of reference frame $\mathcal{A}$ in reference frame $\mathcal{F}$ is given as

$${}^{\mathcal{F}}\boldsymbol{\omega}^{\mathcal{A}} = \dot{\theta}\mathbf{e}_z \tag{4.9}$$

The velocity of point P in reference frame $\mathcal{F}$ is then given as

$$ {}^{\mathcal{F}}\mathbf{v}_P = \frac{{}^{\mathcal{F}}d}{dt}(\mathbf{r}_O) + \frac{{}^{\mathcal{F}}d}{dt}(\mathbf{r}_{P/O}) = {}^{\mathcal{F}}\mathbf{v}_O + {}^{\mathcal{F}}\mathbf{v}_{P/O} \tag{4.10} $$

Now we already have ${}^{\mathcal{F}}\mathbf{v}_O$ from Eq. (4.6). Next, since $\mathbf{r}_{P/O}$ is expressed in the basis $\{\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_z\}$ and $\{\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_z\}$ rotates with angular velocity ${}^{\mathcal{F}}\boldsymbol{\omega}^{\mathcal{A}}$, we can apply the rate of change transport theorem to $\mathbf{r}_{P/O}$ between reference frame $\mathcal{A}$ and reference frame $\mathcal{F}$ as

$$ {}^{\mathcal{F}}\mathbf{v}_{P/O} = \frac{{}^{\mathcal{F}}d}{dt}(\mathbf{r}_{P/O}) = \frac{{}^{\mathcal{A}}d}{dt}(\mathbf{r}_{P/O}) + {}^{\mathcal{F}}\boldsymbol{\omega}^{\mathcal{A}} \times \mathbf{r}_{P/O} \tag{4.11} $$

Now we have that

$$ \frac{{}^{\mathcal{A}}d}{dt}(\mathbf{r}_{P/O}) = \mathbf{0} \tag{4.12} $$

$$ {}^{\mathcal{F}}\boldsymbol{\omega}^{\mathcal{A}} \times \mathbf{r}_{P/O} = \dot{\theta}\mathbf{e}_z \times l\mathbf{e}_r = l\dot{\theta}\mathbf{e}_\theta \tag{4.13} $$

Adding Eq. (4.12) and Eq. (4.13) gives

$$ {}^{\mathcal{F}}\mathbf{v}_{P/O} = l\dot{\theta}\mathbf{e}_\theta \tag{4.14} $$

Therefore, the velocity of the particle in reference frame $\mathcal{F}$ is given as

$$ {}^{\mathcal{F}}\mathbf{v}_P = \dot{x}\mathbf{E}_x + l\dot{\theta}\mathbf{e}_\theta \tag{4.15} $$

Next, the acceleration of point P in reference frame $\mathcal{F}$ is obtained as

$$ {}^{\mathcal{F}}\mathbf{a}_P = \frac{{}^{\mathcal{F}}d}{dt}\left({}^{\mathcal{F}}\mathbf{v}_P\right) \tag{4.16} $$

Now we have that

$$ {}^{\mathcal{F}}\mathbf{v}_P = {}^{\mathcal{F}}\mathbf{v}_O + {}^{\mathcal{F}}\mathbf{v}_{P/O} \tag{4.17} $$

where we have from Eq. (4.6) and Eq. (4.14) that

$$ {}^{\mathcal{F}}\mathbf{v}_O = \dot{x}\mathbf{E}_x \tag{4.18} $$

$$ {}^{\mathcal{F}}\mathbf{v}_{P/O} = l\dot{\theta}\mathbf{e}_\theta \tag{4.19} $$

Consequently,

$$ {}^{\mathcal{F}}\mathbf{a}_P = \frac{{}^{\mathcal{F}}d}{dt}\left({}^{\mathcal{F}}\mathbf{v}_O\right) + \frac{{}^{\mathcal{F}}d}{dt}\left({}^{\mathcal{F}}\mathbf{v}_{P/O}\right) \tag{4.20} $$

Now we already have ${}^{\mathcal{F}}\mathbf{a}_O$ from Eq. (4.7. Furthermore, since ${}^{\mathcal{F}}\mathbf{v}_{P/O}$ is expressed in the basis $\{\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_z\}$ and $\{\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_z\}$ rotates with angular velocity ${}^{\mathcal{F}}\boldsymbol{\omega}^{\mathcal{A}}$, we can obtain ${}^{\mathcal{F}}\mathbf{a}_{P/O}$ by applying the rate of change transport theorem between reference frame $\mathcal{A}$ and reference frame $\mathcal{F}$ as

$$ {}^{\mathcal{F}}\mathbf{a}_{P/O} = \frac{{}^{\mathcal{F}}d}{dt}\left({}^{\mathcal{F}}\mathbf{v}_{P/O}\right) = \frac{{}^{\mathcal{A}}d}{dt}\left({}^{\mathcal{F}}\mathbf{v}_{P/O}\right) + {}^{\mathcal{F}}\boldsymbol{\omega}^{\mathcal{A}} \times {}^{\mathcal{F}}\mathbf{v}_{P/O} \tag{4.21} $$

Now we have that

$$\frac{{}^{\mathcal{A}}d}{dt}\left({}^{\mathcal{F}}\mathbf{v}_{P/O}\right) = l\ddot{\theta}\mathbf{e}_\theta \quad (4.22)$$

$$ {}^{\mathcal{F}}\boldsymbol{\omega}^{\mathcal{A}} \times {}^{\mathcal{F}}\mathbf{v}_{P/O} = \dot{\theta}\mathbf{e}_z \times l\dot{\theta}\mathbf{e}_\theta = -l\dot{\theta}^2\mathbf{e}_r \quad (4.23)$$

Adding Eq. (4.22) and Eq. (4.23) gives

$$ {}^{\mathcal{F}}\mathbf{a}_{P/O} = -l\dot{\theta}^2\mathbf{e}_r + l\ddot{\theta}\mathbf{e}_\theta \quad (4.24)$$

Then, adding Eq. (4.7) and Eq. (4.24), we obtain the acceleration of point P in reference frame $\mathcal{F}$ as

$$ {}^{\mathcal{F}}\mathbf{a}_P = \ddot{x}\mathbf{E}_x - l\dot{\theta}^2\mathbf{e}_r + l\ddot{\theta}\mathbf{e}_\theta \quad (4.25)$$

The acceleration of the center of mass of the system is then computed using the expressions for ${}^{\mathcal{F}}\mathbf{a}_O$ and ${}^{\mathcal{F}}\mathbf{a}_P$ from Eqs. (4.7) and (4.25), respectively. In particular, we have that

$$ {}^{\mathcal{F}}\bar{\mathbf{a}} = \frac{M\,{}^{\mathcal{F}}\mathbf{a}_O + m\,{}^{\mathcal{F}}\mathbf{a}_P}{M+m} \quad (4.26)$$

Substituting the results of Eqs. (4.7) and (4.25) into Eq. (4.26), we obtain

$$ {}^{\mathcal{F}}\bar{\mathbf{a}} = \frac{M\ddot{x}\mathbf{E}_x + m(\ddot{x}\mathbf{E}_x - l\dot{\theta}^2\mathbf{e}_r + l\ddot{\theta}\mathbf{e}_\theta)}{M+m} = \ddot{x}\mathbf{E}_x + \frac{m}{M+m}\left(-l\dot{\theta}^2\mathbf{e}_r + l\ddot{\theta}\mathbf{e}_\theta\right) \quad (4.27)$$

Kinetics

The free body diagram of the block is shown in Fig. 4-1.

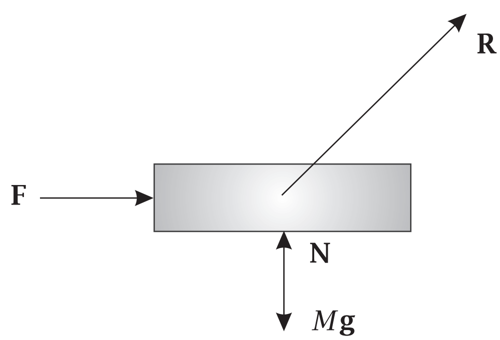

Figure 4-1 Free Body Diagram of Block for Question. 4–1

Using Fig. 4-1, the forces acting on the block are given as

$$\begin{aligned} \mathbf{F} &= \text{External Force} \\ \mathbf{R} &= \text{Reaction Force of Particle on Block} \\ \mathbf{N} &= \text{Reaction Force of Ground on Block} \\ M\mathbf{g} &= \text{Force of Gravity} \end{aligned}$$

Now we have that

$$\begin{aligned}\mathbf{F} &= F\mathbf{E}_x &(4.28)\\ \mathbf{R} &= R\mathbf{e}_r &(4.29)\\ \mathbf{N} &= N\mathbf{E}_y &(4.30)\\ M\mathbf{g} &= Mg\mathbf{E}_y &(4.31)\end{aligned}$$

Then the resultant force acting on the block is given as

$$\mathbf{F}_O = \mathbf{F} + \mathbf{R} + \mathbf{N} + M\mathbf{g} = F\mathbf{E}_x + N\mathbf{E}_y + R\mathbf{e}_r + Mg\mathbf{E}_y \quad (4.32)$$

Using the expression for $\mathbf{e}_r$ from Eq. (4.3), we have that

$$\mathbf{F}_O = F\mathbf{E}_x + N\mathbf{E}_y + R(\sin\theta\mathbf{E}_x - \cos\theta\mathbf{E}_y) + Mg\mathbf{E}_y = (F + R\sin\theta)\mathbf{E}_x + (N - R\cos\theta + Mg)\mathbf{E}_y \quad (4.33)$$

Setting $\mathbf{F}_O$ equal to $M^{\mathcal{F}}\mathbf{a}_O$, we obtain the following two scalar equations:

$$\begin{aligned}F + R\sin\theta &= M\ddot{x} &(4.34)\\ N - R\cos\theta + Mg &= 0 &(4.35)\end{aligned}$$

The free body diagram of the block-particle system is shown in Fig. 4-2.

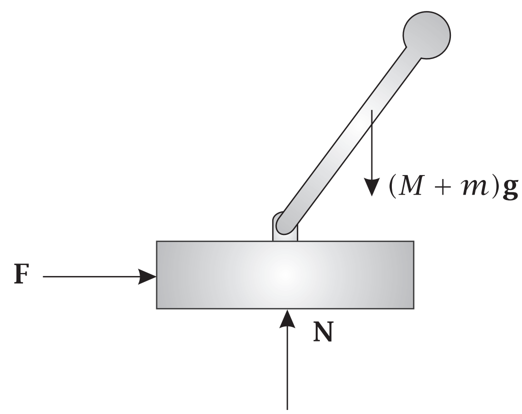

Figure 4-2 Free Body Diagram of Block-Particle System for Question. 4–1

Using Fig. 4-2, the forces acting on the block-particle system are given as

$\mathbf{F}$	=	External Force
$\mathbf{N}$	=	Reaction Force of Ground on Block
$(M+m)\mathbf{g}$	=	Force of Gravity

Now we have that

$$\begin{aligned}\mathbf{F} &= F\mathbf{E}_x &(4.36)\\ \mathbf{N} &= N\mathbf{E}_y &(4.37)\\ (M+m)\mathbf{g} &= (M+m)g\mathbf{E}_y &(4.38)\end{aligned}$$

Therefore, the resultant force acting on the block-particle system is given as

$$\mathbf{F}_T = \mathbf{F} + \mathbf{N} + (M+m)\mathbf{g} = F\mathbf{E}_x + N\mathbf{E}_y + (M+m)g\mathbf{E}_y \tag{4.39}$$

Eq. (4.39) simplifies to

$$\mathbf{F}_T = F\mathbf{E}_x + \left[N + (M+m)g\right]\mathbf{E}_y \tag{4.40}$$

Setting $\mathbf{F}_T$ equal to $(M+m)^{\mathcal{F}}\bar{\mathbf{a}}$, we obtain

$$F\mathbf{E}_x + \left[N + (M+m)g\right]\mathbf{E}_y = (M+m)\left[\ddot{x} + \frac{m}{M+m}\left(-l\dot{\theta}^2\mathbf{e}_r + l\ddot{\theta}\mathbf{e}_\theta\right)\right] \tag{4.41}$$

Eq. (4.41) can be rewritten as

$$F\mathbf{E}_x + \left[N + (M+m)g\right]\mathbf{E}_y = (M+m)\ddot{x} + m\left(-l\dot{\theta}^2\mathbf{e}_r + l\ddot{\theta}\mathbf{e}_\theta\right) \tag{4.42}$$

Substituting the expressions for $\mathbf{e}_r$ and $\mathbf{e}_\theta$ from Eqs. (4.3) and (4.4) into Eq. (4.42), we obtain

$$\begin{aligned} F\mathbf{E}_x + \left[N + (M+m)g\right]\mathbf{E}_y = {} & (M+m)\ddot{x} + m\Big[-l\dot{\theta}^2(\sin\theta\,\mathbf{E}_x - \cos\theta\,\mathbf{E}_y) \\ & +l\ddot{\theta}(\cos\theta\,\mathbf{E}_x + \sin\theta\,\mathbf{E}_y)\Big] \end{aligned} \tag{4.43}$$

Eq. (4.43) simplifies to

$$\begin{aligned} F\mathbf{E}_x + \left[N + (M+m)g\right]\mathbf{E}_y = {} & \left[(M+m)\ddot{x} - ml\dot{\theta}^2\sin\theta + ml\ddot{\theta}\cos\theta\right]\mathbf{E}_x \\ & + (ml\dot{\theta}^2\cos\theta + ml\ddot{\theta}\sin\theta)\mathbf{E}_y \end{aligned} \tag{4.44}$$

Equating components in Eq. (4.44), we obtain the following two scalar equations:

$$F = (M+m)\ddot{x} - ml\dot{\theta}^2\sin\theta + ml\ddot{\theta}\cos\theta \tag{4.45}$$

$$N + (M+m)g = ml\dot{\theta}^2\cos\theta + ml\ddot{\theta}\sin\theta \tag{4.46}$$

System of Two Differential Equations

The first differential equation is Eq. (4.45), i.e.,

$$F = (M+m)\ddot{x} - ml\dot{\theta}^2\sin\theta + ml\ddot{\theta}\cos\theta \tag{4.47}$$

The second differential equation is obtained by using Eqs. (4.34), (4.35) and (4.46). Solving Eq. (4.46) for N, we obtain

$$N = ml\dot{\theta}^2\cos\theta + ml\ddot{\theta}\sin\theta - (M+m)g \tag{4.48}$$

Substituting N from Eq. (4.48) into Eq. (4.35) gives

$$ml\dot{\theta}^2\cos\theta + ml\ddot{\theta}\sin\theta - (M+m)g - R\cos\theta + Mg = 0 \tag{4.49}$$

Eq. (4.49) simplifies to

$$ml\dot{\theta}^2\cos\theta + ml\ddot{\theta}\sin\theta - mg - R\cos\theta = 0 \tag{4.50}$$

Then, multiplying Eq. (4.50) by $\sin\theta$, we obtain

$$ml\dot{\theta}^2\cos\theta\sin\theta + ml\ddot{\theta}\sin^2\theta - mg\sin\theta - R\sin\theta = 0 \tag{4.51}$$

Next, multiplying Eq. (4.34) by $\cos\theta$, we obtain

$$F\cos\theta + R\sin\theta\cos\theta = M\ddot{x}\cos\theta \tag{4.52}$$

Rearranging Eq. (4.52) gives

$$F\cos\theta + R\sin\theta\cos\theta - M\ddot{x}\cos\theta = 0 \tag{4.53}$$

Adding Eq. (4.51) and Eq. (4.53), we have that

$$ml\dot{\theta}^2\cos\theta\sin\theta + ml\ddot{\theta}\sin^2\theta - M\ddot{x}\cos\theta - mg\sin\theta + F\cos\theta = 0 \tag{4.54}$$

Rearranging Eq. (4.54), we obtain the second differential equation of motion as

$$M\ddot{x}\cos\theta - ml\ddot{\theta}\sin^2\theta - ml\dot{\theta}^2\cos\theta\sin\theta + mg\sin\theta = -F\cos\theta \tag{4.55}$$

The system of two differential equations is then given from Eqs. (4.47) and (4.55) as

$$(M+m)\ddot{x} - ml\dot{\theta}^2\sin\theta + ml\ddot{\theta}\cos\theta = F \tag{4.56}$$

$$M\ddot{x}\cos\theta - ml\ddot{\theta}\sin^2\theta - ml\dot{\theta}^2\cos\theta\sin\theta + mg\sin\theta = F\cos\theta \tag{4.57}$$

The system in Eqs. (4.56) and (4.57) can be written in a slightly more elegant form as follows. Multiplying Eq. (4.56) by $\cos\theta$ and subtracting the result from Eq. (4.57), we obtain

$$m\ddot{x}\cos\theta + ml\ddot{\theta} + mg\sin\theta = 0 \tag{4.58}$$

An alternate system of differential equations is then obtained from Eqs. 4.56) and (4.58) as

$$(M+m)\ddot{x} - ml\dot{\theta}^2\sin\theta + ml\ddot{\theta}\cos\theta = F \tag{4.59}$$

$$m\ddot{x}\cos\theta + ml\ddot{\theta} + mg\sin\theta = 0 \tag{4.60}$$

Solution to Question 4–2

First, let $\mathcal{F}$ be a fixed reference frame. Then, choose the following coordinate system fixed in reference frame $\mathcal{F}$:

$$\begin{array}{c}\text{Origin at Location of Block}\\ \mathbf{E}_x \text{ To The Left}\\ \mathbf{E}_z \text{ Into Page}\\ \mathbf{E}_y = \mathbf{E}_z \times \mathbf{E}_x\end{array} \tag{4.61}$$

Now in order to solve this problem we need to apply the principle of linear impulse and momentum to the system and/or subsystems of the system. For this problem, it is convenient to choose the following systems:

- System 1: Ball Bearing
- System 2: Block
- System 3: Bullet and Ball Bearing

Application of Linear Impulse and Linear Momentum to Ball Bearing

Applying linear impulse and momentum to the ball bearing, we have that

$$\hat{\mathbf{F}}_m = m\,{}^{\mathcal{F}}\mathbf{v}'_m - m\,{}^{\mathcal{F}}\mathbf{v}_m \tag{4.62}$$

Now the free body Diagram of the ball bearing is shown in Fig. 4-3 where

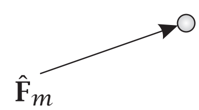

Figure 4-3 Free Body Diagram of Ball Bearing for Question 4–2.

$$\hat{\mathbf{F}}_m = \text{Impulse Applied on Block on Ball Bearing}$$

Now we have that

$$\begin{aligned}{}^{\mathcal{F}}\mathbf{v}_m &= \mathbf{v}_0 = v_0\cos\theta\,\mathbf{E}_x - v_0\sin\theta\,\mathbf{E}_y\\ {}^{\mathcal{F}}\mathbf{v}'_m &= v'_{mx}\mathbf{E}_x + v'_{my}\mathbf{E}_y\end{aligned} \tag{4.63}$$

Also, let

$$\hat{\mathbf{F}}_m \equiv \hat{\mathbf{F}} = \hat{F}_x\mathbf{E}_x + \hat{F}_y\mathbf{E}_y \tag{4.64}$$

Then, substituting the results of Eq. (4.63) and the result of Eq. (4.64) into Eq. (4.62), we obtain

$$m(v_0\cos\theta\,\mathbf{E}_x - v_0\sin\theta\,\mathbf{E}_y) + \hat{F}_x\mathbf{E}_x + \hat{F}_y\mathbf{E}_y = m(v'_{mx}\mathbf{E}_x + v'_{my}\mathbf{E}_y) \tag{4.65}$$

Rearranging, we have

$$(mv_0 \cos\theta + \hat{F}_x)\mathbf{E}_x + (\hat{F}_y - mv_0 \sin\theta)\mathbf{E}_y = mv'_{mx}\mathbf{E}_x + mv'_{my}\mathbf{E}_y \qquad (4.66)$$

We then obtain the following two scalar equations:

$$\begin{aligned} mv_0 \cos\theta + \hat{F}_x &= mv'_{mx} & (4.67) \\ \hat{F}_y - mv_0 \sin\theta &= mv'_{my} & (4.68) \end{aligned}$$

Application of Linear Impulse and Linear Momentum to the Entire System

For the entire system we have that

$$\hat{\mathbf{F}} = {}^{\mathcal{F}}\mathbf{G}' - {}^{\mathcal{F}}\mathbf{G} \qquad (4.69)$$

where ${}^{\mathcal{F}}\mathbf{G}$ and ${}^{\mathcal{F}}\mathbf{G}'$ are the linear momenta of the system before and after impact. Now we have that

$$\begin{aligned} {}^{\mathcal{F}}\mathbf{G} &= m\,{}^{\mathcal{F}}\mathbf{v}_m + M\,{}^{\mathcal{F}}\mathbf{v}_M & (4.70) \\ {}^{\mathcal{F}}\mathbf{G}' &= m\,{}^{\mathcal{F}}\mathbf{v}'_m + M\,{}^{\mathcal{F}}\mathbf{v}'_M & (4.71) \end{aligned}$$

We already have expressions for ${}^{\mathcal{F}}\mathbf{v}_m$ and ${}^{\mathcal{F}}\mathbf{v}'_m$ from Eq. (4.63). In addition, we have that

$$\begin{aligned} {}^{\mathcal{F}}\mathbf{v}_M &= \mathbf{0} \\ {}^{\mathcal{F}}\mathbf{v}'_M &= v'_{Mx}\mathbf{E}_x + v'_{My}\mathbf{E}_y \end{aligned} \qquad (4.72)$$

Next, the free body diagram of the entire system is shown in Fig. 4-4 It is seen

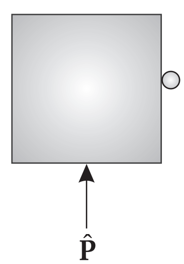

Figure 4-4 Free Body Diagram of Entire System for Question 4–2.

that the only impulse applied to the entire system is $\hat{\mathbf{P}}$ where $\hat{\mathbf{P}}$ is the impulse applied by the ground on the system. Therefore,

$$\hat{\mathbf{F}} = \hat{\mathbf{P}} = \hat{P}\mathbf{E}_y \qquad (4.73)$$

Substituting the results of Eq. (4.63), Eq. (4.72), and Eq. (4.73) into Eq. (4.69), we obtain

$$\hat{P}\mathbf{E}_y = m(v'_{mx}\mathbf{E}_x + v'_{my}\mathbf{E}_y) + M(v'_{Mx}\mathbf{E}_x + v'_{My}\mathbf{E}_y) - m(v_0 \cos\theta\mathbf{E}_x - v_0 \sin\theta\mathbf{E}_y) \qquad (4.74)$$

Rearranging Eq. (4.74), we obtain

$$mv_0\cos\theta\,\mathbf{E}_x + (\hat{P} - mv_0\sin\theta)\mathbf{E}_y = (mv'_{mx} + Mv'_{Mx})\mathbf{E}_x + (mv'_{my} + Mv'_{My})\mathbf{E}_y \tag{4.75}$$

We then obtain the following two scalar equations:

$$mv_0\cos\theta = mv'_{mx} + Mv'_{Mx} \tag{4.76}$$

$$\hat{P} - mv_0\sin\theta = mv'_{my} + Mv'_{My} \tag{4.77}$$

Kinematic Constraints

We know that, because the bullet embeds itself in the block immediately after impact the velocity of the bullet and the block must be the same. Therefore,

$${}^{\mathcal{F}}\mathbf{v}'_m = {}^{\mathcal{F}}\mathbf{v}'_M = {}^{\mathcal{F}}\mathbf{v}' \tag{4.78}$$

For convenience, let

$${}^{\mathcal{F}}\mathbf{v}' = v'_x\mathbf{E}_x + v'_y\mathbf{E}_y \tag{4.79}$$

Second, we know that the bullet and the block must move horizontally after impact. Therefore,

$$v'_y = 0 \tag{4.80}$$

Solving for the Impulse Exerted By Block on Ball Bearing During Impact

The impulse exerted by the block on the ball bearing can be solved for using the results of Eq. (4.67), Eq. (4.68), Eq. (4.76), and Eq. (4.77). First, it is convenient to solve for the post-impact velocity of the block and ball bearing. Because the post-impact velocity of the block and the ball bearing are the same, we have from Eq. (4.76) that

$$mv_0\cos\theta = mv'_{mx} + Mv'_{Mx} = (m+M)v'_x \tag{4.81}$$

Solving this last equation for v'_x, we obtain

$$v'_x = \frac{m}{m+M}v_0\cos\theta \tag{4.82}$$

Then from Eq. (4.67) we have that

$$mv_0\cos\theta + \hat{F}_x = mv'_{mx} = mv'_x = m\frac{m}{m+M}v_0\cos\theta \tag{4.83}$$

Solving this last equation for $\hat{F}_x$, we obtain

$$\hat{F}_x = mv_0\cos\theta\frac{m}{m+M} - mv_0\cos\theta \tag{4.84}$$

Simplifying this last expression, we have that

$$\hat{F}_x = -mv_0 \cos\theta \frac{M}{m+M} \tag{4.85}$$

Also, from Eq. (4.68) we have that

$$\hat{F}_y - mv_0 \sin\theta = mv'_{my} = mv'_y = 0 \tag{4.86}$$

Solving this last equation for $\hat{F}_y$, we obtain

$$\hat{F}_y = mv_0 \sin\theta \tag{4.87}$$

Consequently, the impulse exerted by the block on the bullet is given as

$$\hat{\mathbf{F}}_m = \hat{\mathbf{F}} = mv_0 \cos\theta \frac{M}{m+M}\mathbf{E}_x + mv_0 \sin\theta\, \mathbf{E}_y \tag{4.88}$$

Post-Impact Velocity of Block and Ball Bearing

As stated above, the post-impact velocity of the block and the ball bearing are the same. From Eq. (4.80) and Eq. (4.82) we have that

$$^{\mathcal{F}}\mathbf{v}' = \frac{m}{m+M} v_0 \cos\theta\, \mathbf{E}_x \tag{4.89}$$

Consequently,

$$\begin{aligned} ^{\mathcal{F}}\mathbf{v}'_m &= \frac{m}{m+M} v_0 \cos\theta\, \mathbf{E}_x \\ ^{\mathcal{F}}\mathbf{v}'_M &= \frac{m}{m+M} v_0 \cos\theta\, \mathbf{E}_x \end{aligned} \tag{4.90}$$

Question 4–3

A block of mass m is dropped from a height h above a plate of mass M as shown in Fig. P4-3. The plate is supported by three linear springs, each with spring constant K, and is initially in static equilibrium. Assuming that the compression of the springs due to the weight of the plate is negligible, that the impact is perfectly inelastic, that the block strikes the vertical center of the plate, and that gravity acts downward, determine (a) the velocity of the block and plate immediately after impact and (b) the maximum compression, $x_{\max}$, attained by the springs after impact.

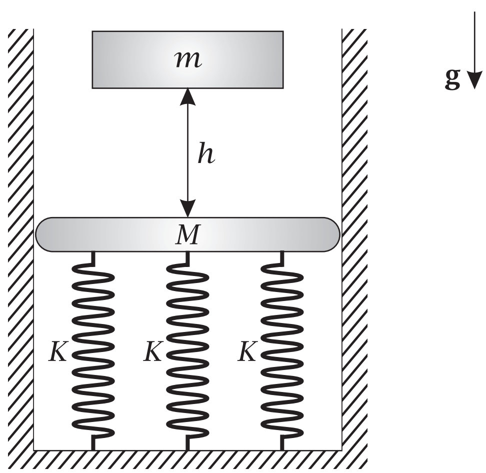

Figure P4-2

Solution to Question 4–3

Kinematics

Let $\mathcal{F}$ be a fixed reference frame. Then, choose the following coordinate system fixed in reference frame $\mathcal{F}$:

$$\begin{array}{ccc} & \text{Origin at } m \text{ at } t=0 & \\ \mathbf{E}_x & = & \text{Down} \\ \mathbf{E}_z & = & \text{Out of Page} \\ \mathbf{E}_y & = & \mathbf{E}_z \times \mathbf{E}_x \end{array}$$

Then the velocity of the block and plate in reference frame $\mathcal{F}$ are given, respectivley, as

$$ {}^{\mathcal{F}}\mathbf{v}_1 = v_1 \mathbf{E}_x \tag{4.91}$$

$$ {}^{\mathcal{F}}\mathbf{v}_2 = v_2 \mathbf{E}_x \tag{4.92}$$

The velocity of the center of mass of the block-plate system in reference frame $\mathcal{F}$ is then given as

$$ {}^{\mathcal{F}}\bar{\mathbf{v}} = \frac{m{}^{\mathcal{F}}\mathbf{v}_1 + M{}^{\mathcal{F}}\mathbf{v}_2}{m+M} == \frac{mv_1 + Mv_2}{m+M}\mathbf{E}_x \tag{4.93} $$

Kinetics

Phase 1: During Descent of Block

The only force acting on the block during its descent is that of gravity. Since gravity is conservative, we know that energy must be conserved. Consequently,

$$ {}^{\mathcal{F}}E(t_0) = {}^{\mathcal{F}}E(t_1) \tag{4.94} $$

where t_0 and t_1 are the times of the start and end of the descent. Now we have that

$$ {}^{\mathcal{F}}E = {}^{\mathcal{F}}T + {}^{\mathcal{F}}U \tag{4.95} $$

Since the block is dropped from rest, we have that

$$ {}^{\mathcal{F}}T(t_0) = 0 \tag{4.96} $$

Next, the kinetic energy of the block at the end of the descent is given as

$$ {}^{\mathcal{F}}T(t_0) = \frac{1}{2}m{}^{\mathcal{F}}\mathbf{v}_1(t_1)\cdot\frac{1}{2}m{}^{\mathcal{F}}\mathbf{v}_1(t_1) = \frac{1}{2}mv_1^2(t_1) \tag{4.97} $$

Also, the initial potential energy is given as

$$ {}^{\mathcal{F}}U(t_0) = -m\mathbf{g}\cdot\mathbf{r}_1(t_0) \tag{4.98} $$

where $\mathbf{r}_1(t_0)$ is the position of the block at time $t = t_0$. Now since the origin of the coordinate system is located at the block at $t = 0$, we know that $\mathbf{r}(t_0) = \mathbf{0}$. Consequently,

$$ {}^{\mathcal{F}}U(t_0) = 0 \tag{4.99} $$

Finally, the potential energy at the end of the descent is given as

$$ {}^{\mathcal{F}}U(t_1) = -m\mathbf{g}\cdot\mathbf{r}_1(t_1) \tag{4.100} $$

Now since the block is dropped from a height h above the plate, we have that

$$ \mathbf{r}_1(t_1) = h\mathbf{E}_x \tag{4.101} $$

Noting that $\mathbf{g} = g\mathbf{E}_x$, we have that

$$ {}^{\mathcal{F}}U(t_1) = -mg\mathbf{E}_x\cdot h\mathbf{E}_x = -mgh \tag{4.102} $$

Equating the energy of the block at times t_0 and t_1 we have that

$$0 = \frac{1}{2} m v_1^2(t_1) - mgh \tag{4.103}$$

Solving Eq. (4.103) for $v_1(t_1)$, we obtain

$$v_1(t_1) = \sqrt{2gh} \tag{4.104}$$

From Eq. (4.104), the velocity of the block in reference frame $\mathcal{F}$ at the end of the descent is given as

$$^{\mathcal{F}}\mathbf{v}_1(t_1) = \sqrt{2gh}\mathbf{E}_x \tag{4.105}$$

Phase 2: During Impact

Applying the principle of linear impulse and linear momentum to the entire system during impact, we have that

$$\hat{\mathbf{F}} = {}^{\mathcal{F}}\mathbf{G}(t_1^+) - {}^{\mathcal{F}}\mathbf{G}(t_1^-) \tag{4.106}$$

Now, because the impact is assumed to occur instantaneously and neither gravity nor the spring can apply an instantaneous impulse, during impact there are no external impulses applied to the system, i.e.,

$$\hat{\mathbf{F}} = \mathbf{0} \tag{4.107}$$

Therefore,

$$^{\mathcal{F}}\mathbf{G}(t_1^+) = {}^{\mathcal{F}}\mathbf{G}(t_1^-) \tag{4.108}$$

Now the linear momentum of the system is given as

$$^{\mathcal{F}}\mathbf{G} = (m + M)\,{}^{\mathcal{F}}\bar{\mathbf{v}} \tag{4.109}$$

Using the expression for ${}^{\mathcal{F}}\bar{\mathbf{v}}$ from Eq. (4.93), we obtain

$$^{\mathcal{F}}\mathbf{G} = (m + M)\frac{mv_1 + Mv_2}{m + M}\mathbf{E}_x = (mv_1 + Mv_2)\mathbf{E}_x \tag{4.110}$$

Then, noting that the plate is motionless before impact and using the expression for ${}^{\mathcal{F}}\mathbf{v}_1(t_1)$ from Eq. (4.105), we have that

$$^{\mathcal{F}}\mathbf{G}(t_1^-) = m\sqrt{2gh}\mathbf{E}_x \tag{4.111}$$

Consequently, we have that

$$(m + M)\,{}^{\mathcal{F}}\bar{\mathbf{v}}(t_1^+) = m\sqrt{2gh}\mathbf{E}_x \tag{4.112}$$

Solving for ${}^{\mathcal{F}}\bar{\mathbf{v}}(t_1^+)$, we obtain the velocity of the center of mass of the system the instant after impact as

$$ {}^{\mathcal{F}}\bar{\mathbf{v}}(t_1^+) = \frac{m}{m+M}\sqrt{2gh}\mathbf{E}_x \tag{4.113} $$

Next, we know that the impact is perfectly inelastic. Consequently, the coefficient of restitution is zero, i.e., $e = 0$. Now we have the coefficient of restitution condition as

$$ e = \frac{{}^{\mathcal{F}}\mathbf{v}_2' \cdot \mathbf{n} - {}^{\mathcal{F}}\mathbf{v}_1' \cdot \mathbf{n}}{{}^{\mathcal{F}}\mathbf{v}_1 \cdot \mathbf{n} - {}^{\mathcal{F}}\mathbf{v}_2 \cdot \mathbf{n}} \tag{4.114} $$

Now since $e = 0$, Eq. (4.114) becomes

$$ \frac{{}^{\mathcal{F}}\mathbf{v}_2(t_1^+) \cdot \mathbf{n} - {}^{\mathcal{F}}\mathbf{v}_1(t_1^+) \cdot \mathbf{n}}{{}^{\mathcal{F}}\mathbf{v}_1(t_1^-) \cdot \mathbf{n} - {}^{\mathcal{F}}\mathbf{v}_2(t_1^-) \cdot \mathbf{n}} = 0 \tag{4.115} $$

Eq. (4.115) implies that

$$ {}^{\mathcal{F}}\mathbf{v}_2(t_1^+) \cdot \mathbf{n} - {}^{\mathcal{F}}\mathbf{v}_1(t_1^+) \cdot \mathbf{n} = 0 \tag{4.116} $$

Now for this problem is it seen that the direction of impact is along $\mathbf{E}_x$, i.e., $\mathbf{n} = \mathbf{E}_x$. Furthermore, since the velocities of the block and the plate are also along $\mathbf{E}_x$, we have that

$$ {}^{\mathcal{F}}\mathbf{v}_1(t_1^+) \cdot \mathbf{n} = v_1(t_1^+) \tag{4.117} $$

$$ {}^{\mathcal{F}}\mathbf{v}_2(t_1^+) \cdot \mathbf{n} = v_2(t_1^+) \tag{4.118} $$

Eq. (4.116) then reduces to

$$ v_2(t_1^+) - v_1(t_1^+) = 0 \tag{4.119} $$

Eq. (4.119) implies that

$$ v_2(t_1^+) = v_1(t_1^+) \tag{4.120} $$

It is seen from Eq. (4.120) that the post-impact velocities of the block and plate are the same. Now we have that

$$ (m+M)^{\mathcal{F}}\bar{\mathbf{v}} = m^{\mathcal{F}}\mathbf{v}_1 + M^{\mathcal{F}}\mathbf{v}_2 \tag{4.121} $$

Consequently, using the expression for ${}^{\mathcal{F}}\bar{\mathbf{v}}(t_1^+)$ from Eq. (4.113), we obtain

$$ (m+M)^{\mathcal{F}}\bar{\mathbf{v}}(t_1^+) = (m+M)\frac{m\sqrt{2gh}}{m+M}\mathbf{E}_x = m\sqrt{2gh}\mathbf{E}_x = mv_1(t_1^+)\mathbf{E}_x + Mv_2(t_1^+) \tag{4.122} $$

But since $v_1(t_1^+) = v_2(t_1^+) \equiv v(t_1^+)$, we have that

$$ m\sqrt{2gh} = (m+M)v(t_1^+) \tag{4.123} $$

Solving for $v(t_1^+)$ gives

$$v(t_1^+) = \frac{m}{m+M}\sqrt{2gh}\mathbf{E}_x \tag{4.124}$$

Therefore, the post-impact velocities of the block and plate in reference frame $\mathcal{F}$ are given as

$$\begin{aligned} {}^{\mathcal{F}}\mathbf{v}_1(t_1^+) &= \frac{m}{m+M}\sqrt{2gh}\mathbf{E}_x & (4.125)\\ {}^{\mathcal{F}}\mathbf{v}_2(t_1^+) &= \frac{m}{m+M}\sqrt{2gh}\mathbf{E}_x & (4.126)\end{aligned}$$

Phase 3: Maximum Compression of Spring After Impact

First, it is important to recognize that, because the impact between the block and the plate was perfectly inelastic, the block and plate will move together after impact, i.e., the block and plate will move as if they are a single body after impact. Next, for this part of the problem we see that the only forces acting on the block and plate are those of the three springs. Since the attachment points of the springs are inertially fixed (in this case they are fixed in reference frame $\mathcal{F}$), we know that the spring forces will be conservative. Thus, the total energy of the system after impact will be conserved. Now the total energy is given as

$${}^{\mathcal{F}}E = {}^{\mathcal{F}}T + {}^{\mathcal{F}}U \tag{4.127}$$

Now, the velocity of the block and plate is given as

$${}^{\mathcal{F}}\mathbf{v} = v\mathbf{E}_x \tag{4.128}$$

Therefore, the kinetic energy of the system is given as

$${}^{\mathcal{F}}T = \frac{1}{2}(m+M)\,{}^{\mathcal{F}}\mathbf{v}\cdot{}^{\mathcal{F}}\mathbf{v} = \frac{1}{2}(m+M)v^2 \tag{4.129}$$

Next, the potential energy is due to the three springs and is given as

$${}^{\mathcal{F}}U_s = 3\left[\frac{1}{2}K(\ell-\ell_0)^2\right] = \frac{3}{2}K(\ell-\ell_0)^2 \tag{4.130}$$

Now since the springs are initially uncompressed, we have that

$$\ell(t_1) = \ell_0 \tag{4.131}$$

Now since the springs are compressing in this phase, we know that $\ell - \ell_0$ will be less than zero for $t > t_1$. In order to account for the fact that $\ell - \ell_0 < 0$, let $x_{\max} = \ell_0 - \ell(t_2)$. Then the potential energy in the springs at the point of maximum compression is given as

$${}^{\mathcal{F}}U_s(t_2) = 3\left[\frac{1}{2}K(\ell(t_2)-\ell_0)^2\right] = \frac{3}{2}Kx_{\max}^2 \tag{4.132}$$

Then the total potential energy at the point of maximum compression is given as

$$ {}^{\mathcal{F}}U(t_2) = -(m+M)g(h+x_{\max}) + \frac{3}{2}Kx_{\max}^2 \qquad (4.133) $$

where we note that the potential energy due to gravity at t_2 is different from the potential energy due to gravity at t_1 because the position of the particle and block has changed from $-h\mathbf{E}_x$ to $-(h+x_{\max})\mathbf{E}_x$ in moving from t_1 to t_2. Now, using the results from phase 2, we know at t_1 that

$$ \begin{aligned} {}^{\mathcal{F}}T(t_1) &= \frac{1}{2}(m+M)\left(\frac{m}{m+M}\sqrt{2gh}\right)^2 = \frac{m^2gh}{m+M} && (4.134) \\ {}^{\mathcal{F}}U(t_1) &= -(m+M)gh && (4.135) \end{aligned} $$

Then the total energy of the system at t_1 is given as

$$ {}^{\mathcal{F}}E(t_1) = {}^{\mathcal{F}}T(t_1) + {}^{\mathcal{F}}U(t_1) = 0 \qquad (4.136) $$

Furthermore, at t_2 we have that

$$ \begin{aligned} {}^{\mathcal{F}}T(t_2) &= 0 && (4.137) \\ {}^{\mathcal{F}}U(t_2) &= -(m+M)g(h+x_{\max}) + \frac{3}{2}Kx_{\max}^2 && (4.138) \end{aligned} $$

where we note that the kinetic energy at t_2 is zero because t_2 is the time at which the springs have attained their maximum compression. Then the total energy of the system at t_2 is given as

$$ {}^{\mathcal{F}}E(t_2) = {}^{\mathcal{F}}T(t_2) + {}^{\mathcal{F}}U(t_2) = -(m+M)g(h+x_{\max}) + \frac{3}{2}Kx_{\max}^2 \qquad (4.139) $$

Setting ${}^{\mathcal{F}}E(t_2)$ equal to ${}^{\mathcal{F}}E(t_1)$, we have that

$$ -(m+M)g(h+x_{\max}) + \frac{3}{2}Kx_{\max}^2 = 0 \qquad (4.140) $$

Eq. (4.140) can be simplified to

$$ \frac{3}{2}Kx_{\max}^2 - (m+M)gx_{\max} - (m+M)gh = 0 \qquad (4.141) $$

Simplifying Eq. (4.141) further, we obtain

$$ x_{\max}^2 - \frac{2(m+M)g}{3K}x_{\max} - \frac{2(m+M)gh}{3K} = 0 \qquad (4.142) $$

Eq. (4.142) is a quadratic in $x_{\max}$. The roots of Eq. (4.142) are given from the quadratic formula as

$$ x_{\max} = \frac{\dfrac{2(m+M)g}{3K} \pm \sqrt{\left(\dfrac{2(m+M)g}{3K}\right)^2 + \dfrac{8(m+M)gh}{3K}}}{2} \qquad (4.143) $$

For simplicity, let

$$a = \frac{2(m+M)g}{3K} \tag{4.144}$$

Then Eq. (4.143) can be written as

$$x_{\max} = \frac{a \pm \sqrt{a^2 + 4ah}}{2} \tag{4.145}$$

Eq. (4.145) can be rewritten as

$$x_{\max} = \frac{a \pm a\sqrt{1 + 4h/a}}{2} = \frac{a}{2}\left[1 \pm \sqrt{1 + 4h/a}\right] \tag{4.146}$$

Now since h and a are positive, we know that the quantity $1 + 4h/a$ must be greater than unity. Consequently, one of the roots of Eq. (4.146) must be negative. Choosing the positive root, we obtain

$$x_{\max} = \frac{a}{2}\left[1 + \sqrt{1 + 4h/a}\right] \tag{4.147}$$

Substituting the expression for a from Eq. (4.144), we obtain the maximum compression of the springs after impact as

$$x_{\max} = \frac{(m+M)g}{3K}\left[1 + \sqrt{1 + \frac{6Kh}{(m+M)g}}\right] \tag{4.148}$$

Question 4–8

A particle of mass m_A slides without friction along a fixed vertical rigid rod. The particle is attached via a rigid massless arm to a particle of mass m_B where m_B slides without friction along a fixed horizontal rigid rod. Assuming that θ is the angle between the horizontal rod and the arm and that gravity acts downward, determine the differential equation of motion for the system in terms of the angle θ.

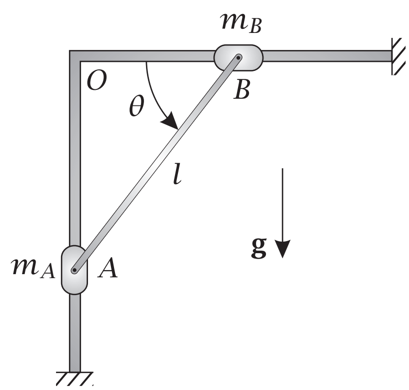

Figure P4-3

Solution to Question 4–8

Before beginning the kinematics for this problem, it is important to know which balance laws will be used. Since this problem consists of two particles, it is convenient to use the following balance laws: (1) Newton's 2^{nd} law and (2) a moment balance. Since it is generally convenient to choose the center of mass as a reference point, the moment balance will be performed relative to the center of mass. Consequently, we will need to compute the following kinematic quantities: (1) the acceleration of the center of mass and (2) the rate of change of angular momentum of the system relative to the center of mass.

Kinematics

Let $\mathcal{F}$ be a fixed reference frame. Then, choose the following coordinate system fixed in reference frame $\mathcal{F}$:

$$\begin{array}{ccc} & \text{Origin at Point } O & \\ \mathbf{E}_x & = & \text{Along } OA \\ \mathbf{E}_y & = & \text{Along } OB \\ \mathbf{E}_z & = & \mathbf{E}_x \times \mathbf{E}_y \end{array}$$

Then the position of each particle is given in terms of the basis $\{\mathbf{E}_x, \mathbf{E}_y, \mathbf{E}_z\}$ as

$$\begin{aligned} \mathbf{r}_A &= l\sin\theta\,\mathbf{E}_x && (4.149)\\ \mathbf{r}_B &= l\cos\theta\,\mathbf{E}_y && (4.150) \end{aligned}$$

The position of the center of mass of the system is then given as

$$\bar{\mathbf{r}} = \frac{m_A\mathbf{r}_A + m_B\mathbf{r}_B}{m_A + m_B} \tag{4.151}$$

Substituting the expressions for $\mathbf{r}_A$ and $\mathbf{r}_B$ from Eq. (4.149) and Eq. (4.150), respectively, into Eq. (4.151), we obtain

$$\bar{\mathbf{r}} = \frac{m_A l\sin\theta\,\mathbf{E}_x + m_B l\cos\theta\,\mathbf{E}_y}{m_A + m_B} \tag{4.152}$$

Computing the rates of change of $\mathbf{r}_A$ and $\mathbf{r}_B$ in reference frame $\mathcal{F}$, we obtain

$$\begin{aligned} {}^{\mathcal{F}}\mathbf{v}_A &= \frac{{}^{\mathcal{F}}d}{dt}(\mathbf{r}_A) = l\dot\theta\cos\theta\,\mathbf{E}_x && (4.153)\\ {}^{\mathcal{F}}\mathbf{v}_B &= \frac{{}^{\mathcal{F}}d}{dt}(\mathbf{r}_A) = -l\dot\theta\sin\theta\,\mathbf{E}_y && (4.154) \end{aligned}$$

Computing the rates of change of ${}^{\mathcal{F}}\mathbf{v}_A$ and ${}^{\mathcal{F}}\mathbf{v}_B$ in reference frame $\mathcal{F}$, we obtain

$$\begin{aligned} {}^{\mathcal{F}}\mathbf{a}_A &= \frac{{}^{\mathcal{F}}d}{dt}\left({}^{\mathcal{F}}\mathbf{v}_A\right) = l(\ddot\theta\cos\theta - \dot\theta^2\sin\theta)\mathbf{E}_x && (4.155)\\ {}^{\mathcal{F}}\mathbf{a}_B &= \frac{{}^{\mathcal{F}}d}{dt}\left({}^{\mathcal{F}}\mathbf{v}_B\right) = -l(\ddot\theta\sin\theta + \dot\theta^2\cos\theta)\mathbf{E}_y && (4.156) \end{aligned}$$

The acceleration of the center of mass is then given as

$${}^{\mathcal{F}}\bar{\mathbf{a}} = \frac{m_A{}^{\mathcal{F}}\mathbf{a}_A + m_B{}^{\mathcal{F}}\mathbf{a}_B}{m_A + m_B} \tag{4.157}$$

Substituting the results of Eq. (4.155) and Eq. (4.156) into Eq. (4.157), we obtain the acceleration of the center of mass in reference frame $\mathcal{F}$ as

$${}^{\mathcal{F}}\bar{\mathbf{a}} = \frac{m_A l(\ddot\theta\cos\theta - \dot\theta^2\sin\theta)\mathbf{E}_x - m_B l(\ddot\theta\sin\theta + \dot\theta^2\cos\theta)\mathbf{E}_y}{m_A + m_B} \tag{4.158}$$

Next, since we will eventually be performing a moment balance relative to the center of mass, we will need the rate of change of angular momentum relative to the center of mass. Recalling the expression for the rate of change of angular momentum relative to the center of mass for a system of particles, we have that

$$\frac{{}^{\mathcal{F}}d}{dt}\left({}^{\mathcal{F}}\bar{\mathbf{H}}\right) = (\mathbf{r}_A - \bar{\mathbf{r}}) \times m_A({}^{\mathcal{F}}\mathbf{a}_A - {}^{\mathcal{F}}\bar{\mathbf{a}}) + (\mathbf{r}_B - \bar{\mathbf{r}}) \times m_B({}^{\mathcal{F}}\mathbf{a}_B - {}^{\mathcal{F}}\bar{\mathbf{a}}) \tag{4.159}$$

Now from Eqs. (4.149), (4.150), and (4.152) we have

$$\begin{aligned} \mathbf{r}_A - \bar{\mathbf{r}} &= l\sin\theta\mathbf{E}_x - \frac{m_A l\sin\theta\mathbf{E}_x + m_B l\cos\theta\mathbf{E}_y}{m_A + m_B} \\ &= \frac{m_B l\sin\theta\mathbf{E}_x - m_B l\cos\theta\mathbf{E}_y}{m_A + m_B} \end{aligned} \tag{4.160}$$

$$\begin{aligned} \mathbf{r}_B - \bar{\mathbf{r}} &= l\cos\theta\mathbf{E}_y - \frac{m_A l\sin\theta\mathbf{E}_x + m_B l\cos\theta\mathbf{E}_y}{m_A + m_B} \\ &= \frac{-m_A l\sin\theta\mathbf{E}_x + m_A l\cos\theta\mathbf{E}_y}{m_A + m_B} \end{aligned} \tag{4.161}$$

Similarly, from Eqs. (4.155), (4.156), and (4.158) we have

$${}^{\mathcal{F}}\mathbf{a}_A - {}^{\mathcal{F}}\bar{\mathbf{a}} = \frac{m_B l(\ddot\theta\sin\theta + \dot\theta^2\cos\theta)\mathbf{E}_x + m_B l(\ddot\theta\cos\theta - \dot\theta^2\sin\theta)\mathbf{E}_y}{m_A + m_B} \tag{4.162}$$

$${}^{\mathcal{F}}\mathbf{a}_B - {}^{\mathcal{F}}\bar{\mathbf{a}} = \frac{-m_A l(\ddot\theta\sin\theta + \dot\theta^2\cos\theta)\mathbf{E}_x - m_A l(\ddot\theta\cos\theta - \dot\theta^2\sin\theta)\mathbf{E}_y}{m_A + m_B} \tag{4.163}$$

The rate of change of angular momentum relative to the center of mass is then given as

$$\begin{aligned} \frac{{}^{\mathcal{F}}d}{dt}\left({}^{\mathcal{F}}\bar{\mathbf{H}}\right) = {} & \frac{m_B l\cos\theta\mathbf{E}_x - m_B l\sin\theta\mathbf{E}_y}{m_A + m_B} \\ & \times m_A\left(\frac{m_B l(\ddot\theta\sin\theta + \dot\theta^2\cos\theta)\mathbf{E}_x + m_B l(\ddot\theta\cos\theta - \dot\theta^2\sin\theta)\mathbf{E}_y}{m_A + m_B}\right) \\ & + \frac{-m_A l\cos\theta\mathbf{E}_x + m_A l\sin\theta\mathbf{E}_y}{m_A + m_B} \\ & \times m_B\left(\frac{-m_A l(\ddot\theta\sin\theta + \dot\theta^2\cos\theta)\mathbf{E}_x - m_A l(\ddot\theta\cos\theta - \dot\theta^2\sin\theta)\mathbf{E}_y}{m_A + m_B}\right) \end{aligned} \tag{4.164}$$

Simplifying Eq. (4.164), we obtain

$$\frac{{}^{\mathcal{F}}d}{dt}\left({}^{\mathcal{F}}\bar{\mathbf{H}}\right) = \frac{(m_A m_B^2 + m_B m_A^2)l^2\ddot\theta}{(m_A + m_B)^2}\mathbf{E}_z \tag{4.165}$$

which further simplifies to

$$\frac{{}^{\mathcal{F}}d}{dt}\left({}^{\mathcal{F}}\bar{\mathbf{H}}\right) = \frac{m_A m_B}{m_A + m_B}l^2\ddot\theta\mathbf{E}_z \tag{4.166}$$

Kinetics

The free body diagram of the system is shown in Fig. 4-5. It can be seen from

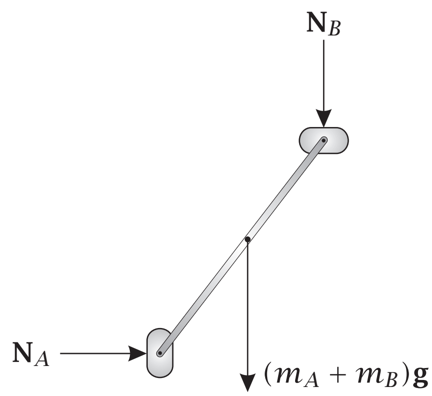

Figure 4-5 Free Body Diagram of System for Question 4–17.

the free body diagram that the following three forces act on the system:

$$\begin{aligned}\mathbf{N}_A &= \text{Reaction Force of Vertical Track on } m_A \\ \mathbf{N}_B &= \text{Reaction Force of Horizontal Track on } m_B \\ (m_A + m_B)\mathbf{g} &= \text{Force of Gravity at Center of Mass}\end{aligned}$$

Now we know that $\mathbf{N}_A$ and $\mathbf{N}_B$ act orthogonal to the horizontal and vertical tracks, respectively, i.e.,

$$\mathbf{N}_A = N_A\mathbf{E}_y \tag{4.167}$$

$$\mathbf{N}_B = N_B\mathbf{E}_x \tag{4.168}$$

Furthermore, the force of gravity acts vertically downward, i.e.,

$$(m_A + m_B)\mathbf{g} = (m_A + m_B)g\mathbf{E}_x \tag{4.169}$$

Then the total force acting on the system is given as

$$\mathbf{F} = \mathbf{N}_A + \mathbf{N}_B + (m_A + m_B)\mathbf{g} = N_A\mathbf{E}_y + N_B\mathbf{E}_x + (m_A + m_B)g\mathbf{E}_x \tag{4.170}$$

Eq. (4.170) simplifies to

$$\mathbf{F} = [N_B + (m_A + m_B)g]\mathbf{E}_x + N_A\mathbf{E}_y \tag{4.171}$$

Then, using the expression for ${}^{\mathcal{F}}\bar{\mathbf{a}}$ from Eq. (4.158), we have that

$$m\,{}^{\mathcal{F}}\bar{\mathbf{a}} = m_A l(\ddot{\theta}\cos\theta - \dot{\theta}^2\sin\theta)\mathbf{E}_x - m_B l(\ddot{\theta}\sin\theta + \dot{\theta}^2\cos\theta)\mathbf{E}_y \tag{4.172}$$

Applying Newton's 2^{nd} law, we obtain

$$\begin{aligned}[N_B + (m_A + m_B)g]\mathbf{E}_x + N_A\mathbf{E}_y = {} & m_A l(\ddot{\theta}\cos\theta - \dot{\theta}^2\sin\theta)\mathbf{E}_x \\ & - m_B l(\ddot{\theta}\sin\theta + \dot{\theta}^2\cos\theta)\mathbf{E}_y\end{aligned} \tag{4.173}$$

Equating components, we obtain the following two scalar equations:

$$N_B + (m_A + m_B)g = m_A l(\ddot{\theta}\cos\theta - \dot{\theta}^2\sin\theta) \tag{4.174}$$

$$N_A = -m_B l(\ddot{\theta}\sin\theta + \dot{\theta}^2\cos\theta) \tag{4.175}$$

Next, since gravity passes through the center of mass, the moment due to all forces relative to the center of mass is due to $\mathbf{N}_A$ and $\mathbf{N}_B$ and is given as

$$\bar{\mathbf{M}} = (\mathbf{r}_A - \bar{\mathbf{r}}) \times \mathbf{N}_A + (\mathbf{r}_B - \bar{\mathbf{r}}) \times \mathbf{N}_B \tag{4.176}$$

Substituting the results of Eqs. (4.160), (4.161), (4.161), (4.167), and (4.168) into Eq. (4.176), we obtain

$$\bar{\mathbf{M}} = \left(\frac{m_B l\sin\theta\mathbf{E}_x - m_B l\cos\theta\mathbf{E}_y}{m_A + m_B}\right) \times N_A\mathbf{E}_y + \left(\frac{-m_A l\sin\theta\mathbf{E}_y + m_A l\cos\theta\mathbf{E}_y}{m_A + m_B}\right) \times N_B\mathbf{E}_x \tag{4.177}$$

Eq. (4.177) simplifies to

$$\bar{\mathbf{M}} = \frac{m_B N_A l\sin\theta - m_A N_B l\cos\theta}{m_A + m_B}\mathbf{E}_z \tag{4.178}$$

Setting $\bar{\mathbf{M}}$ in Eq. (4.178) equal to ${}^{\mathcal{F}}d({}^{\mathcal{F}}\bar{\mathbf{H}})/dt$ using the expression for ${}^{\mathcal{F}}d({}^{\mathcal{F}}\bar{\mathbf{H}})/dt$ from Eq. (4.166), we obtain

$$\frac{m_B N_A l\sin\theta - m_A N_B l\cos\theta}{m_A + m_B}\mathbf{E}_z = \frac{m_A m_B}{m_A + m_B} l^2\ddot{\theta}\mathbf{E}_z \tag{4.179}$$

Eq. (4.179) simplifies to

$$m_B N_A l\sin\theta - m_A N_B l\cos\theta = m_A m_B l^2\ddot{\theta} \tag{4.180}$$

We can now use Eqs. (4.174), (4.175), and (4.180) to obtain the differential equation of motion. First, multiplying Eq. (4.174) by $m_A l\cos\theta$ and multiplying Eq. (4.175) by $m_B l\sin\theta$, we obtain

$$m_A N_B l\cos\theta + m_A(m_A + m_B)gl\cos\theta = m_A^2 l^2(\ddot{\theta}\cos\theta - \dot{\theta}^2\sin\theta)\cos\theta \tag{4.181}$$

$$m_B N_A l\sin\theta = -m_B^2 l^2(\ddot{\theta}\sin\theta + \dot{\theta}^2\cos\theta)\sin\theta \tag{4.182}$$

Subtracting Eq. (4.181) from (4.182), we obtain

$$\begin{aligned} m_B N_A l\sin\theta - m_A N_B l\cos\theta - m_A(m_A + m_B)gl\cos\theta \\ = -m_B^2 l^2(\ddot{\theta}\sin\theta + \dot{\theta}^2\cos\theta)\sin\theta \\ - m_A^2 l^2(\ddot{\theta}\cos\theta - \dot{\theta}^2\sin\theta)\cos\theta \end{aligned} \tag{4.183}$$

Then, substituting $m_B N_A l\sin\theta - m_A N_B l\cos\theta$ from Eq. (4.180) into Eq. (4.183), we obtain

$$\begin{aligned} m_A m_B l^2\ddot{\theta} - m_A(m_A + m_B)gl\cos\theta = -m_B^2 l^2(\ddot{\theta}\sin\theta + \dot{\theta}^2\cos\theta)\sin\theta \\ - m_A^2 l^2(\ddot{\theta}\cos\theta - \dot{\theta}^2\sin\theta)\cos\theta \end{aligned} \tag{4.184}$$

Rearranging this last result, we obtain

$$\begin{aligned}(m_A m_B + m_A^2\cos^2\theta + m_B^2\sin^2\theta)\ddot{\theta} + (m_A^2 - m_B^2)\dot{\theta}^2\cos\theta\sin\theta \\ - m_A(m_A+m_B)\frac{g}{l}\cos\theta = 0\end{aligned} \tag{4.185}$$

Now we note that

$$m_A m_B + m_A^2\cos^2\theta + m_B^2\sin^2\theta = (m_A\cos^2\theta + m_B\sin^2\theta)(m_A+m_B) \tag{4.186}$$

Substituting Eq. (4.186) into Eq. (4.185), we obtain

$$\begin{aligned}(m_A\cos^2\theta + m_B\sin^2\theta)(m_A+m_B)\ddot{\theta} + (m_A^2 - m_B^2)\dot{\theta}^2\cos\theta\sin\theta \\ - m_A(m_A+m_B)\frac{g}{l}\cos\theta = 0\end{aligned} \tag{4.187}$$

Furthermore, we have that

$$(m_A^2 - m_B^2) = (m_A - m_B)(m_A + m_B) \tag{4.188}$$

Substituting Eq. (4.188) into Eq. (4.187), we obtain

$$\begin{aligned}(m_A\cos^2\theta + m_B\sin^2\theta)(m_A+m_B)\ddot{\theta} + (m_A - m_B)(m_A+m_B)\dot{\theta}^2\cos\theta\sin\theta \\ - m_A(m_A+m_B)\frac{g}{l}\cos\theta = 0\end{aligned} \tag{4.189}$$

Finally, observing that $m_A + m_B$ is common to all of the terms in Eq. (4.189), we obtain the differential equation of motion as

$$(m_A\cos^2\theta + m_B\sin^2\theta)\ddot{\theta} + (m_A - m_B)\dot{\theta}^2\cos\theta\sin\theta - m_A\frac{g}{l}\cos\theta = 0 \tag{4.190}$$

Question 4–17

A dumbbell consists of two particles A and B each of mass m connected by a rigid massless rod of length $2l$. Each end of the dumbbell slides without friction along a fixed circular track of radius R as shown in Fig. P4-17. Knowing that θ is the angle from the vertical to the center of the rod and that gravity acts downward, determine the differential equation of motion for the dumbbell.

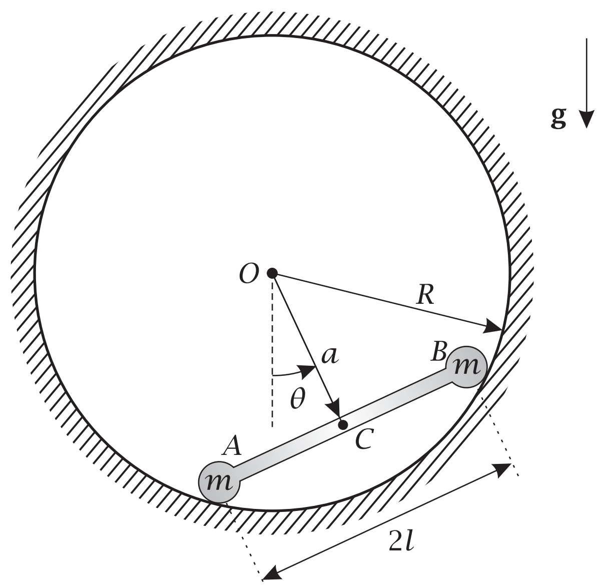

Figure P4-12

Solution to Question 4–17

This problem can be solved using many different approaches. For the purposes of this solution, we will show the following three two most prominent of these approaches: (1) using point O as the reference point or (2) using the center of mass as the reference point. In order to apply all of these approaches, we will need to compute the following kinematic quantities: (a) the acceleration of each particle, the angular momentum of the system relative to point O, and (c) the angular momentum of the system relative to the center of mass.

Kinematics

First, let $\mathcal{F}$ be a fixed reference frame. Then, choose the following coordinate system fixed in reference frame $\mathcal{F}$:

$$
\begin{array}{lcl}
 & \text{Origin at Point } O & \\
\mathbf{E}_x & = & \text{Along } OC \text{ When } \theta = 0 \\
\mathbf{E}_z & = & \text{Out of Page} \\
\mathbf{E}_y & = & \mathbf{E}_z \times \mathbf{E}_x
\end{array}
$$

Next, let $\mathcal{A}$ be a reference frame that rotates with the dumbbell. Then, choose the following coordinate system fixed in reference frame $\mathcal{A}$:

$$
\begin{array}{lcl}
 & \text{Origin at Point } O & \\
\mathbf{e}_r & = & \text{Along } OC \\
\mathbf{e}_z & = & \mathbf{E}_z (= \text{ Out of Page }) \\
\mathbf{e}_\theta & = & \mathbf{e}_z \times \mathbf{e}_r
\end{array}
$$

The geometry of the bases $\{\mathbf{E}_x, \mathbf{E}_y, \mathbf{E}_z\}$ and $\{\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_z\}$ is shown in Fig. 4-6 from which we have that

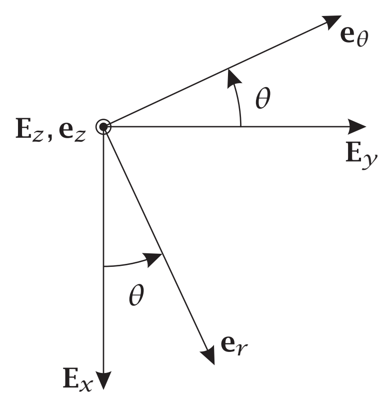

Figure 4-6 Geometry of Bases $\{\mathbf{E}_x, \mathbf{E}_y, \mathbf{E}_z\}$ and $\{\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_z\}$ for 4–17.

$$
\begin{aligned}
\mathbf{E}_x &= \cos\theta\,\mathbf{e}_r - \sin\theta\,\mathbf{e}_\theta && (4.191) \\
\mathbf{E}_y &= -\sin\theta\,\mathbf{e}_r + \cos\theta\,\mathbf{e}_\theta && (4.192)
\end{aligned}
$$

It is seen that the angular velocity of reference frame $\mathcal{A}$ in reference frame $\mathcal{F}$ is given as

$$
{}^{\mathcal{F}}\boldsymbol{\omega}^{\mathcal{A}} = \dot{\theta}\mathbf{e}_z \tag{4.193}
$$

Furthermore, the position of each particle is then given in terms of the basis $\{\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_z\}$ as

$$
\begin{aligned}
\mathbf{r}_A &= a\mathbf{e}_r - l\mathbf{e}_\theta && (4.194) \\
\mathbf{r}_B &= a\mathbf{e}_r + l\mathbf{e}_\theta && (4.195)
\end{aligned}
$$

The velocity of each particle can then be obtained using the transport theorem as

$$
\begin{aligned}
{}^{\mathcal{F}}\mathbf{v}_A &= \frac{{}^{\mathcal{A}}d\mathbf{r}_A}{dt} + {}^{\mathcal{F}}\boldsymbol{\omega}^{\mathcal{A}} \times \mathbf{r}_A && (4.196) \\
{}^{\mathcal{F}}\mathbf{v}_B &= \frac{{}^{\mathcal{A}}d\mathbf{r}_B}{dt} + {}^{\mathcal{F}}\boldsymbol{\omega}^{\mathcal{A}} \times \mathbf{r}_B && (4.197)
\end{aligned}
$$

Now since a and l are constant, we have that

$$
\begin{aligned}
\frac{{}^{\mathcal{A}}d\mathbf{r}_A}{dt} &= \mathbf{0} && (4.198) \\
\frac{{}^{\mathcal{A}}d\mathbf{r}_B}{dt} &= \mathbf{0} && (4.199)
\end{aligned}
$$

Furthermore,

$$
\begin{aligned}
{}^{\mathcal{F}}\boldsymbol{\omega}^{\mathcal{A}} \times \mathbf{r}_A &= \dot{\theta}\mathbf{e}_z \times (a\mathbf{e}_r - l\mathbf{e}_\theta) = l\dot{\theta}\mathbf{e}_r + a\dot{\theta}\mathbf{e}_\theta && (4.200) \\
{}^{\mathcal{F}}\boldsymbol{\omega}^{\mathcal{A}} \times \mathbf{r}_B &= \dot{\theta}\mathbf{e}_z \times (a\mathbf{e}_r + l\mathbf{e}_\theta) = -l\dot{\theta}\mathbf{e}_r + a\dot{\theta}\mathbf{e}_\theta && (4.201)
\end{aligned}
$$

The velocity of each particle is then given as

$$
\begin{aligned}
{}^{\mathcal{F}}\mathbf{v}_A &= l\dot{\theta}\mathbf{e}_r + a\dot{\theta}\mathbf{e}_\theta && (4.202) \\
{}^{\mathcal{F}}\mathbf{v}_A &= -l\dot{\theta}\mathbf{e}_r + a\dot{\theta}\mathbf{e}_\theta && (4.203)
\end{aligned}
$$

The acceleration of each particle is then obtained from the transport theorem as

$$
\begin{aligned}
{}^{\mathcal{F}}\mathbf{a}_A &= \frac{{}^{\mathcal{A}}d}{dt}\left({}^{\mathcal{F}}\mathbf{v}_A\right) + {}^{\mathcal{F}}\boldsymbol{\omega}^{\mathcal{A}} \times {}^{\mathcal{F}}\mathbf{v}_A && (4.204) \\
{}^{\mathcal{F}}\mathbf{a}_B &= \frac{{}^{\mathcal{A}}d}{dt}\left({}^{\mathcal{F}}\mathbf{v}_B\right) + {}^{\mathcal{F}}\boldsymbol{\omega}^{\mathcal{A}} \times {}^{\mathcal{F}}\mathbf{v}_B && (4.205)
\end{aligned}
$$

Now we have that

$$
\begin{aligned}
\frac{{}^{\mathcal{A}}d}{dt}\left({}^{\mathcal{F}}\mathbf{v}_A\right) &= l\ddot{\theta}\mathbf{e}_r + a\ddot{\theta}\mathbf{e}_\theta && (4.206) \\
\frac{{}^{\mathcal{A}}d}{dt}\left({}^{\mathcal{F}}\mathbf{v}_B\right) &= -l\ddot{\theta}\mathbf{e}_r + a\ddot{\theta}\mathbf{e}_\theta && (4.207)
\end{aligned}
$$

Furthermore,

$$
\begin{aligned}
{}^{\mathcal{F}}\boldsymbol{\omega}^{\mathcal{A}} \times {}^{\mathcal{F}}\mathbf{v}_A &= \dot{\theta}\mathbf{e}_z \times (l\dot{\theta}\mathbf{e}_r + a\dot{\theta}\mathbf{e}_\theta) = -a\dot{\theta}^2\mathbf{e}_r + l\dot{\theta}^2\mathbf{e}_\theta && (4.208) \\
{}^{\mathcal{F}}\boldsymbol{\omega}^{\mathcal{A}} \times {}^{\mathcal{F}}\mathbf{v}_B &= \dot{\theta}\mathbf{e}_z \times (-l\dot{\theta}\mathbf{e}_r + a\dot{\theta}\mathbf{e}_\theta) = -a\dot{\theta}^2\mathbf{e}_r - l\dot{\theta}^2\mathbf{e}_\theta && (4.209)
\end{aligned}
$$

Therefore, the acceleration of each particle is given as

$$
\begin{aligned}
{}^{\mathcal{F}}\mathbf{a}_A &= (l\ddot{\theta} - a\dot{\theta}^2)\mathbf{e}_r + (a\ddot{\theta} + l\dot{\theta}^2)\mathbf{e}_\theta && (4.210) \\
{}^{\mathcal{F}}\mathbf{a}_B &= -(l\ddot{\theta} + a\dot{\theta}^2)\mathbf{e}_r + (a\ddot{\theta} - l\dot{\theta}^2)\mathbf{e}_\theta && (4.211)
\end{aligned}
$$

Using the accelerations obtained in Eq. (4.210) and Eq. (4.211), the angular momentum of the system relative to the (inertially fixed) point O is obtained as

$$^{\mathcal{F}}\mathbf{H}_O = (\mathbf{r}_A - \mathbf{r}_O) \times m_A {}^{\mathcal{F}}\mathbf{v}_A + (\mathbf{r}_B - \mathbf{r}_O) \times m_B {}^{\mathcal{F}}\mathbf{v}_B \tag{4.212}$$

Noting that $\mathbf{r}_O = \mathbf{0}$ and substituting the results from Eq. (4.202) and Eq. (4.203) into Eq. (4.212), we obtain

$$^{\mathcal{F}}\mathbf{H}_O = (a\mathbf{e}_r - l\mathbf{e}_\theta) \times m_A(l\dot{\theta}\mathbf{e}_r + a\dot{\theta}\mathbf{e}_\theta) + (a\mathbf{e}_r + l\mathbf{e}_\theta) \times m_B(-l\dot{\theta}\mathbf{e}_r + a\dot{\theta}\mathbf{e}_\theta) \tag{4.213}$$

Simplifying Eq. (4.213), we obtain

$$^{\mathcal{F}}\mathbf{H}_O = \left[m_A(a^2 + l^2)\dot{\theta} + m_B(a^2 + l^2)\dot{\theta}\right]\mathbf{e}_z \tag{4.214}$$

Finally, noting that $m_A = m_B = m$, we obtain $^{\mathcal{F}}\mathbf{H}_O$ as

$$^{\mathcal{F}}\mathbf{H}_O = 2m(a^2 + l^2)\dot{\theta}\mathbf{e}_z \tag{4.215}$$

Similarly, the angular momentum of the system relative to the center of mass of the system is given as

$$^{\mathcal{F}}\bar{\mathbf{H}} = (\mathbf{r}_A - \bar{\mathbf{r}}) \times m_A({}^{\mathcal{F}}\mathbf{v}_A - {}^{\mathcal{F}}\bar{\mathbf{v}}) + (\mathbf{r}_B - \bar{\mathbf{r}}) \times m_B({}^{\mathcal{F}}\mathbf{v}_B - {}^{\mathcal{F}}\mathbf{v}_A) \tag{4.216}$$

Now we have that

$$\bar{\mathbf{r}} = \frac{m_A\mathbf{r}_A + m_B\mathbf{r}_B}{m_A + m_B} = \frac{m(a\mathbf{e}_r - l\mathbf{e}_\theta) + m(a\mathbf{e}_r + l\mathbf{e}_\theta)}{2m} = a\mathbf{e}_r \tag{4.217}$$

Therefore,

$$\begin{aligned} \mathbf{r}_A - \bar{\mathbf{r}} &= a\mathbf{e}_r - l\mathbf{e}_\theta - a\mathbf{e}_r = -l\mathbf{e}_\theta && (4.218) \\ \mathbf{r}_B - \bar{\mathbf{r}} &= a\mathbf{e}_r + l\mathbf{e}_\theta - a\mathbf{e}_r = l\mathbf{e}_\theta && (4.219) \\ & && (4.220) \end{aligned}$$

Furthermore, from the transport theorem we have that

$$^{\mathcal{F}}\bar{\mathbf{v}} = \frac{^{\mathcal{F}}d\bar{\mathbf{r}}}{dt} = \frac{^{\mathcal{A}}d\bar{\mathbf{r}}}{dt} + {}^{\mathcal{F}}\boldsymbol{\omega}^{\mathcal{A}} \times \bar{\mathbf{r}} \tag{4.221}$$

Now since a is constant, we have that

$$\frac{^{\mathcal{A}}d\bar{\mathbf{r}}}{dt} = \mathbf{0} \tag{4.222}$$

Furthermore,

$$^{\mathcal{F}}\boldsymbol{\omega}^{\mathcal{A}} \times \bar{\mathbf{r}} = \dot{\theta}\mathbf{e}_z \times a\mathbf{e}_r = a\dot{\theta}\mathbf{e}_\theta \tag{4.223}$$

Consequently,

$$^{\mathcal{F}}\bar{\mathbf{v}} = a\dot{\theta}\mathbf{e}_\theta \tag{4.224}$$

We then have that

$$
\begin{aligned}
{}^{\mathcal{F}}\mathbf{v}_A - {}^{\mathcal{F}}\bar{\mathbf{v}} &= l\dot{\theta}\mathbf{e}_r + a\dot{\theta}\mathbf{e}_\theta - a\dot{\theta}\mathbf{e}_\theta = l\dot{\theta}\mathbf{e}_r && (4.225)\\
{}^{\mathcal{F}}\mathbf{v}_B - {}^{\mathcal{F}}\bar{\mathbf{v}} &= -l\dot{\theta}\mathbf{e}_r + a\dot{\theta}\mathbf{e}_\theta - a\dot{\theta}\mathbf{e}_\theta = -l\dot{\theta}\mathbf{e}_r && (4.226)\\
& && (4.227)
\end{aligned}
$$

Substituting the results of Eq. (4.218), Eq. (4.219), Eq. (4.225), Eq. (4.226), and Eq. (4.217) into Eq. (4.216), we obtain

$$
{}^{\mathcal{F}}\bar{\mathbf{H}} = -l\mathbf{e}_\theta \times m_A(l\dot{\theta}\mathbf{e}_r) + l\mathbf{e}_\theta \times m_B(-l\dot{\theta}\mathbf{e}_r) = m_A l^2\dot{\theta}\mathbf{e}_z + m_B l^2\dot{\theta}\mathbf{e}_z = (m_A + m_B)l^2\dot{\theta}\mathbf{e}_z \tag{4.228}
$$

Again, noting that $m_A = m_B = m$, Eq. (4.228) simplifies to

$$
{}^{\mathcal{F}}\bar{\mathbf{H}} = 2ml^2\dot{\theta}\mathbf{e}_z \tag{4.229}
$$

Kinetics

The free body diagram of the system is shown in Fig. 4-7. It can be seen that the

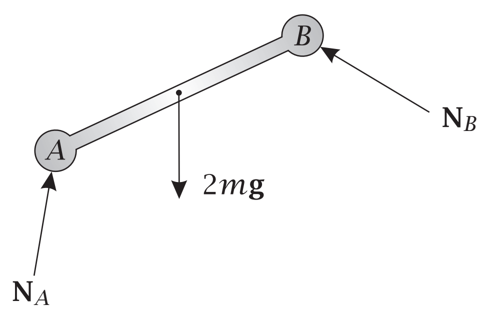

Figure 4-7 Free Body Diagram of System for Question 4–17.

following three forces act on the system:

$$
\begin{aligned}
\mathbf{N}_A &= \text{Reaction Force of Track on Particle A}\\
\mathbf{N}_B &= \text{Reaction Force of Track on Particle B}\\
2m\mathbf{g} &= \text{Force of Gravity}
\end{aligned}
$$

Now we know that $\mathbf{N}_A$ and $\mathbf{N}_B$ must be orthogonal to the track at points A and B, respectively. Consequently, we can express $\mathbf{N}_A$ and $\mathbf{N}_B$ as

$$
\begin{aligned}
\mathbf{N}_A &= N_A\mathbf{u}_A\\
\mathbf{N}_B &= N_B\mathbf{u}_B
\end{aligned} \tag{4.230}
$$

where $\mathbf{u}_A$ and $\mathbf{u}_B$ are the directions orthogonal to the track at points A and B, respectively. Now, since the track is circular, we know that the directions orthogonal to the track at points A and B are along $\mathbf{r}_A$ and $\mathbf{r}_B$, respectively, i.e.,

$$\mathbf{u}_A = \frac{\mathbf{r}_A}{\|\mathbf{r}_A\|} \tag{4.231}$$

$$\mathbf{u}_B = \frac{\mathbf{r}_B}{\|\mathbf{r}_B\|} \tag{4.232}$$

Consequently, we can write $\mathbf{N}_A$ and $\mathbf{N}_B$ as

$$\mathbf{N}_A = N_A \frac{\mathbf{r}_A}{\|\mathbf{r}_A\|} \tag{4.233}$$

$$\mathbf{N}_B = N_B \frac{\mathbf{r}_B}{\|\mathbf{r}_B\|} \tag{4.234}$$

Next, the force due to gravity can be written as

$$2m\mathbf{g} = 2mg\mathbf{E}_x \tag{4.235}$$

where $\mathbf{E}_x$ is the vertically downward direction. Substituting the result for $\mathbf{E}_x$ in terms of $\mathbf{e}_r$ and $\mathbf{e}_\theta$ from Eq. (4.192), we have that

$$2m\mathbf{g} = 2mg(\cos\theta\,\mathbf{e}_r - \sin\theta\,\mathbf{e}_\theta) = 2mg\cos\theta\,\mathbf{e}_r - 2mg\sin\theta\,\mathbf{e}_\theta \tag{4.236}$$

With the resolution of forces completed, we can now proceed to solve the problem using the two approaches stated at the beginning of this solution, namely (1) applying a moment balance using point O as the reference point and (2) applying a force balance and a moment balance using the center of mass as the reference point.

Method 1: Point O as Reference Point

Since O is a point fixed in the inertial reference frame $\mathcal{F}$, we have that

$$\frac{{}^{\mathcal{F}}d}{dt}\left({}^{\mathcal{F}}\mathbf{H}_O\right) = \mathbf{M}_O \tag{4.237}$$

First, differentiating the expression for ${}^{\mathcal{F}}\mathbf{H}_O$ in reference frame $\mathcal{F}$ using ${}^{\mathcal{F}}\mathbf{H}_O$ in reference frame $\mathcal{F}$ from Eq. (4.215), we obtain ${}^{\mathcal{F}}d\left({}^{\mathcal{F}}\mathbf{H}_O\right)/dt$ as

$$\frac{{}^{\mathcal{F}}d}{dt}\left({}^{\mathcal{F}}\mathbf{H}_O\right) = 2m(a^2 + l^2)\ddot{\theta}\mathbf{e}_z \tag{4.238}$$

Next, using the free body diagram of Fig. 4-7), we have have the moment relative to point O as

$$\mathbf{M}_O = (\mathbf{r}_A - \mathbf{r}_O) \times \mathbf{N}_A + (\mathbf{r}_B - \mathbf{r}_O) \times \mathbf{N}_B + (\mathbf{r}_g - \mathbf{r}_O) \times 2m\mathbf{g} \tag{4.239}$$

Now since $\mathbf{r}_O = \mathbf{0}$, Eq. (4.239) can be written as

$$\mathbf{M}_O = \mathbf{r}_A \times \mathbf{N}_A + \mathbf{r}_B \times \mathbf{N}_B + \mathbf{r}_g \times 2m\mathbf{g} \tag{4.240}$$

Now from Eq. (4.233) and Eq. (4.234) we have that the forces $\mathbf{N}_A$ and $\mathbf{N}_B$ lie in the direction of $\mathbf{r}_A$ and $\mathbf{r}_B$, respectively. Consequently, we have that

$$\mathbf{r}_A \times \mathbf{N}_A = \mathbf{0} \tag{4.241}$$
$$\mathbf{r}_B \times \mathbf{N}_B = \mathbf{0} \tag{4.242}$$

Therefore, $\mathbf{M}_O$ reduces to

$$\mathbf{M}_O = \mathbf{r}_g \times 2m\mathbf{g} \tag{4.243}$$

Now we note that gravity acts at the center of mass, i.e., $\mathbf{r}_g = \bar{\mathbf{r}} = a\mathbf{e}_r$. Using this last fact together with the expression for $2m\mathbf{g}$ from Eq. (4.236), we have $\mathbf{M}_O$ as

$$\mathbf{M}_O = a\mathbf{e}_r \times (2mg\cos\theta\,\mathbf{e}_r - 2mg\sin\theta\,\mathbf{e}_\theta) = -2mga\sin\theta\,\mathbf{e}_z \tag{4.244}$$

Equating $\mathbf{M}_O$ from Eq. (4.244) with ${}^{\mathcal{F}}d\left({}^{\mathcal{F}}\mathbf{H}_O\right)/dt$ from Eq. (4.238), we obtain

$$2m(a^2 + l^2)\ddot{\theta}\mathbf{e}_z = -2mga\sin\theta\,\mathbf{e}_z \tag{4.245}$$

Simplifying Eq. (4.245), we obtain the differential equation of motion as

$$\ddot{\theta} + \frac{ag}{a^2 + l^2}\sin\theta = 0 \tag{4.246}$$

Method 2: Center of Mass as Reference Point

Using the center of mass as the reference point, we need to apply a force balance and a moment balance. Applying a force balance, we have that

$$\mathbf{F} = 2m^{\mathcal{F}}\bar{\mathbf{a}} \tag{4.247}$$

Now the resultant force acting on the particle is given as

$$\mathbf{F} = \mathbf{N}_A + \mathbf{N}_B + 2m\mathbf{g} \tag{4.248}$$

Using the expressions for $\mathbf{N}_A$ and $\mathbf{N}_B$ from Eq. (4.233) and Eq. (4.234), respectively, we have $\mathbf{N}_A$ and $\mathbf{N}_B$ as

$$\mathbf{N}_A = N_A\frac{\mathbf{r}_A}{\|\mathbf{r}_A\|} = N_A\frac{a\mathbf{e}_r - l\mathbf{e}_\theta}{\sqrt{a^2 + l^2}} \tag{4.249}$$
$$\mathbf{N}_B = N_B\frac{\mathbf{r}_B}{\|\mathbf{r}_B\|} = N_B\frac{a\mathbf{e}_r + l\mathbf{e}_\theta}{\sqrt{a^2 + l^2}} \tag{4.250}$$

Then, adding the expressions in Eq. (4.249) and Eq. (4.250) to the expression for $2m\mathbf{g}$ from Eq. (4.236), we have that

$$\mathbf{F} = N_A\frac{a\mathbf{e}_r - l\mathbf{e}_\theta}{\sqrt{a^2 + l^2}} + N_B\frac{a\mathbf{e}_r + l\mathbf{e}_\theta}{\sqrt{a^2 + l^2}} + 2mg\cos\theta\,\mathbf{e}_r - 2mg\sin\theta\,\mathbf{e}_\theta \tag{4.251}$$

Simplifying Eq. (4.251), we obtain

$$\mathbf{F} = \left[(N_A + N_B) \frac{a}{\sqrt{a^2 + l^2}} + 2mg\cos\theta \right] \mathbf{e}_r + \left[(N_B - N_A) \frac{l}{\sqrt{a^2 + l^2}} - 2mg\sin\theta \right] \mathbf{e}_\theta \tag{4.252}$$

Now the acceleration of the center of mass is given as

$$^{\mathcal{F}}\bar{\mathbf{a}} = \frac{^{\mathcal{F}}d}{dt}\left(^{\mathcal{F}}\bar{\mathbf{v}}\right) = \frac{^{\mathcal{A}}d}{dt}\left(^{\mathcal{F}}\bar{\mathbf{v}}\right) + {}^{\mathcal{F}}\boldsymbol{\omega}^{\mathcal{A}} \times {}^{\mathcal{F}}\bar{\mathbf{v}} \tag{4.253}$$

Using the expression for $^{\mathcal{F}}\bar{\mathbf{a}}$ from Eq. (4.224), we have that

$$\frac{^{\mathcal{A}}d}{dt}\left(^{\mathcal{F}}\bar{\mathbf{v}}\right) = a\ddot{\theta}\mathbf{e}_\theta \tag{4.254}$$

$$^{\mathcal{F}}\boldsymbol{\omega}^{\mathcal{A}} \times {}^{\mathcal{F}}\bar{\mathbf{v}} = \dot{\theta}\mathbf{e}_z \times a\dot{\theta}\mathbf{e}_\theta = -a\dot{\theta}^2\mathbf{e}_r \tag{4.255}$$

The acceleration of the center of mass in reference frame $\mathcal{F}$ is then given as

$$^{\mathcal{F}}\bar{\mathbf{a}} = -a\dot{\theta}^2\mathbf{e}_r + a\ddot{\theta}\mathbf{e}_\theta \tag{4.256}$$

Setting $\mathbf{F}$ in Eq. (4.252) equal to $2m{}^{\mathcal{F}}\bar{\mathbf{a}}$ using $^{\mathcal{F}}\bar{\mathbf{a}}$ from Eq. (4.256), we obtain

$$\left[(N_A + N_B) \frac{a}{\sqrt{a^2 + l^2}} + 2mg\cos\theta \right] \mathbf{e}_r + \left[(N_B - N_A) \frac{l}{\sqrt{a^2 + l^2}} - 2mg\sin\theta \right] \mathbf{e}_\theta = 2m(-a\dot{\theta}^2\mathbf{e}_r + a\ddot{\theta}\mathbf{e}_\theta) \tag{4.257}$$

Equating components, we obtain the following two scalar equations:

$$(N_A + N_B) \frac{a}{\sqrt{a^2 + l^2}} + 2mg\cos\theta = -2ma\dot{\theta}^2 \tag{4.258}$$

$$(N_B - N_A) \frac{l}{\sqrt{a^2 + l^2}} - 2mg\sin\theta = 2ma\ddot{\theta} \tag{4.259}$$

Next, we need to apply a balance of angular momentum relative to the center of mass, i.e.,

$$\bar{\mathbf{M}} = \frac{^{\mathcal{F}}d}{dt}\left(^{\mathcal{F}}\bar{\mathbf{H}}\right) \tag{4.260}$$

First, using the expression for $^{\mathcal{F}}\bar{\mathbf{H}}$ from Eq. (4.229), we obtain $^{\mathcal{F}}d\left(^{\mathcal{F}}\bar{\mathbf{H}}\right)/dt$

$$\frac{^{\mathcal{F}}d}{dt}\left(^{\mathcal{F}}\bar{\mathbf{H}}\right) = 2ml^2\ddot{\theta}\mathbf{e}_z \tag{4.261}$$

Next, since gravity passes through the center of mass, the moment due to all forces relative to the center of mass of the system is given as

$$\bar{\mathbf{M}} = (\mathbf{r}_A - \bar{\mathbf{r}}) \times \mathbf{N}_A + (\mathbf{r}_B - \bar{\mathbf{r}}) \times \mathbf{N}_B \tag{4.262}$$

Then, using the expressions for $\mathbf{r}_A - \bar{\mathbf{r}}$ and $\mathbf{r}_B - \bar{\mathbf{r}}$ from Eq. (4.218) and Eq. (4.219), respectively, and the expressions for $\mathbf{N}_A$ and $\mathbf{N}_B$ from Eq. (4.249) and Eq. (4.250), respectively, we obtain $\bar{\mathbf{M}}$ as

$$\bar{\mathbf{M}} = -l\mathbf{e}_\theta \times N_A \frac{a\mathbf{e}_r - l\mathbf{e}_\theta}{\sqrt{a^2 + l^2}} + l\dot{\theta}\mathbf{e}_\theta \times N_B \frac{a\mathbf{e}_r + l\mathbf{e}_\theta}{\sqrt{a^2 + l^2}} \tag{4.263}$$

Simplifying Eq. (4.263), we obtain

$$\bar{\mathbf{M}} = (N_A - N_B)\frac{al}{\sqrt{a^2 + l^2}}\mathbf{e}_z \tag{4.264}$$

Then, setting $\bar{\mathbf{M}}$ from Eq. (4.264) equal to ${}^{\mathcal{F}}d\left({}^{\mathcal{F}}\bar{\mathbf{H}}\right)/dt$ from Eq. (4.261), we obtain

$$(N_A - N_B)\frac{al}{\sqrt{a^2 + l^2}}\mathbf{e}_z = 2ml^2\ddot{\theta}\mathbf{e}_z \tag{4.265}$$

We then obtain the following scalar equation:

$$(N_A - N_B)\frac{al}{\sqrt{a^2 + l^2}} = 2ml^2\ddot{\theta} \tag{4.266}$$

Eq. (4.259), Eq. (4.259), and Eq. (4.266) can now be used together to obtain the differential equation. First, multiplying Eq. (4.259) by a, we obtain

$$(N_B - N_A)\frac{al}{\sqrt{a^2 + l^2}} - 2mga\sin\theta = ma^2\ddot{\theta} \tag{4.267}$$

Next, adding Eq. (4.267) to Eq. (4.266), we obtain

$$-2mga\sin\theta = (2ml^2 + 2ma^2)\ddot{\theta} \tag{4.268}$$

Simplifying Eq. (4.268), we obtain the differential equation of motion as

$$\ddot{\theta} + \frac{ag}{a^2 + l^2}\sin\theta = 0 \tag{4.269}$$