Chapter 5

Kinetics of Rigid Bodies

Question 5–1

A homogeneous circular cylinder of mass m and radius r is at rest atop a thin sheet of paper as shown in Fig. P5-1. The paper lies flat on a horizontal surface. Suddenly, the paper is pulled with a very large velocity to the right and is removed from under the cylinder. Assuming that the removal of the paper takes place in a time t, that the coefficient of dynamic friction between all surfaces is μ, and that gravity acts downward, determine (a) the velocity of the center of mass of the cylinder and (b) the angular velocity of the cylinder the instant that the paper is removed.

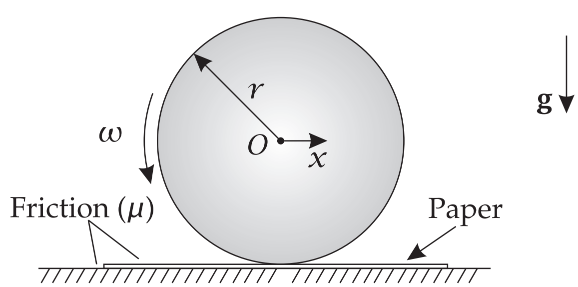

Figure P5-1

Solution to Question 5–1

First, let $\mathcal{F}$ be the ground. Then, it is convenient to choose the following coordinate system fixed in reference frame $\mathcal{F}$:

$$\begin{array}{ccr} & \text{Origin at } O & \\ \mathbf{E}_x & = & \text{To The Right} \\ \mathbf{E}_z & = & \text{Into Page} \\ \mathbf{E}_y & = & \mathbf{E}_z \times \mathbf{E}_x \end{array}$$

Next, let $\mathcal{D}$ be the cylinder. Then, choose the following coordinate system fixed in reference frame $\mathcal{D}$:

$$\begin{array}{ccc} & \text{Origin at } O & \\ \mathbf{e}_r & = & \text{Fixed in } \mathcal{D} \\ \mathbf{e}_z & = & \mathbf{E}_z \\ \mathbf{e}_\theta & = & \mathbf{e}_z \times \mathbf{e}_r \end{array}$$

Now, in order to solve this problem, we need to apply linear impulse and momentum to the center of mass of the cylinder and angular impulse and momentum about the center of mass of the cylinder. In order to apply these two principles, we use the free body diagram shown in Fig. 5-1 where

$$\begin{array}{lcl} \mathbf{N} & = & \text{Reaction Force of Paper (Surface) on Disk} \\ m\mathbf{g} & = & \text{Force of Gravity} \\ \mathbf{F}_f & = & \text{Force of Friction} \end{array}$$

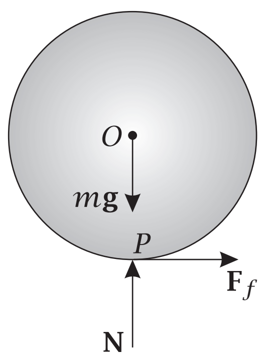

Figure 5-1 Free Body Diagram of Disk

Now from the geometry we have that

$$\begin{aligned} \mathbf{N} &= N\mathbf{E}_y \\ m\mathbf{g} &= mg\mathbf{E}_y \\ \mathbf{F}_f &= -\mu\|\mathbf{N}\|\frac{\mathbf{v}_{\text{rel}}}{\|\mathbf{v}_{\text{rel}}\|} \end{aligned} \tag{5.1}$$

Now we need to determine $\mathbf{v}_{\text{rel}}$. Denoting the point of contact between the disk and the paper by P (see Fig. 5-1), we note that

$$\mathbf{v}_{\text{rel}} = {}^{\mathcal{F}}\mathbf{v}_P - {}^{\mathcal{F}}\mathbf{v}_{\text{paper}} \tag{5.2}$$

where ${}^{\mathcal{F}}\mathbf{v}_{\text{paper}}$ is the velocity of the paper in reference frame $\mathcal{F}$. Since the paper is pulled in the positive $\mathbf{E}_x$-direction, we have that

$${}^{\mathcal{F}}\mathbf{v}_{\text{paper}} = v_{\text{paper}}\mathbf{E}_x \tag{5.3}$$

Next, we need to determine ${}^{\mathcal{F}}\mathbf{v}_P$. From kinematics of rigid bodies we have that

$$ {}^{\mathcal{F}}\mathbf{v}_P - {}^{\mathcal{F}}\mathbf{v}_O = {}^{\mathcal{F}}\boldsymbol{\omega}^{\mathcal{R}} \times (\mathbf{r}_P - \mathbf{r}_O) \tag{5.4} $$

where $\mathcal{R}$ denotes the reference frame of the cylinder and ${}^{\mathcal{F}}\boldsymbol{\omega}^{\mathcal{R}}$ is the angular velocity of reference frame $\mathcal{R}$ in reference frame $\mathcal{F}$. From the geometry we have that

$$ {}^{\mathcal{F}}\boldsymbol{\omega}^{\mathcal{R}} = \omega \mathbf{E}_z \tag{5.5} $$

and

$$ \mathbf{r}_P - \mathbf{r}_O = r\mathbf{E}_y \tag{5.6} $$

Consequently,

$$ {}^{\mathcal{F}}\mathbf{v}_P - {}^{\mathcal{F}}\mathbf{v}_O = \omega \mathbf{E}_z \times r\mathbf{E}_y = -r\omega \mathbf{E}_x \tag{5.7} $$

Furthermore,

$$ \mathbf{r}_O = x\mathbf{E}_x \tag{5.8} $$

which implies that

$$ {}^{\mathcal{F}}\mathbf{v}_O = \frac{{}^{\mathcal{F}}d}{dt}(\mathbf{r}_O) = \dot{x}\mathbf{E}_x \tag{5.9} $$

We then have that

$$ {}^{\mathcal{F}}\mathbf{v}_P = {}^{\mathcal{F}}\mathbf{v}_O - r\omega \mathbf{E}_x = (\dot{x} - r\omega)\mathbf{E}_x \tag{5.10} $$

Now, since the paper is "suddenly" pulled to the right, it implies that the paper is pulled such that its speed is extremely large. Therefore,

$$ v_{\text{paper}} \gg \dot{x} - r\omega \tag{5.11} $$

which implies that

$$ \dot{x} - r\omega - v_{\text{paper}} \ll 0 \tag{5.12} $$

Now we have that

$$ \mathbf{v}_{\text{rel}} = \mathbf{v}_P - \mathbf{v}_{\text{paper}} = (\dot{x} - r\omega)\mathbf{E}_x - v_{\text{paper}}\mathbf{E}_x = (\dot{x} - r\omega - v_{\text{paper}})\mathbf{E}_x \tag{5.13} $$

But from Eq. (5.12) we see that

$$ \mathbf{v}_{\text{rel}} = -|\dot{x} - r\omega - v_{\text{paper}}|\mathbf{E}_x \tag{5.14} $$

which implies that

$$ \frac{\mathbf{v}_{\text{rel}}}{\|\mathbf{v}_{\text{rel}}\|} = -\mathbf{E}_x \tag{5.15} $$

The force of friction is then given as

$$ \mathbf{F}_f = \mu\|\mathbf{N}\|\mathbf{E}_x \tag{5.16} $$

Now that we have expressions for all of the forces, we move on to the application of linear impulse and momentum and angular impulse and momentum

(a) Velocity of Center of Mass of Cylinder at Instant When Paper is Removed

We have that

$$\mathbf{F} = m^{\mathcal{F}}\bar{\mathbf{a}} = m^{\mathcal{F}}\mathbf{a}_O \tag{5.17}$$

Differentiating Eq. (5.9) in reference frame $\mathcal{F}$, we have

$${}^{\mathcal{F}}\mathbf{a}_O = \ddot{x}\mathbf{E}_x \tag{5.18}$$

Furthermore, using the result of the force resolution from above, we have

$$\mathbf{F} = \mathbf{N} + m\mathbf{g} + \mathbf{F}_f = N\mathbf{E}_y + mg\mathbf{E}_y + \mu\|\mathbf{N}\|\mathbf{E}_x = (N + mg)\mathbf{E}_y + \mu\|\mathbf{N}\|\mathbf{E}_x \tag{5.19}$$

Equating $\mathbf{F}$ and $m^{\mathcal{F}}\mathbf{a}_O$, we obtain

$$(N + mg)\mathbf{E}_y + \mu\|\mathbf{N}\|\mathbf{E}_x = m\ddot{x}\mathbf{E}_x \tag{5.20}$$

which yields the following two scalar equations:

$$\begin{aligned} N + mg &= 0 \\ \mu\|\mathbf{N}\| &= m\ddot{x} \end{aligned} \tag{5.21}$$

Therefore,

$$N = -mg \tag{5.22}$$

which implies that

$$\mathbf{N} = -mg\mathbf{E}_y \tag{5.23}$$

which further implies that

$$\|\mathbf{N}\| = mg \tag{5.24}$$

We then have that

$$\mu mg = m\ddot{x} \tag{5.25}$$

This last equation can be integrated from 0 to Δt to give

$$\int_0^{\Delta t} \mu mg dt = \int_0^{\Delta t} m\ddot{x} dt \tag{5.26}$$

We then have that

$$\mu mg\Delta t = m\dot{x}(\Delta t) - m\dot{x}(t = 0) = mv_O(\Delta t) - mv_O(t = 0) \tag{5.27}$$

Noting that the disk is initially stationary, we have that

$$v_O(t = 0) = 0 \tag{5.28}$$

which implies that

$$\mu mg\Delta t mv_O(\Delta t) \tag{5.29}$$

Solving this last equation for $v_O(\Delta t)$, we obtain

$$v_O(\Delta t) = \mu g\Delta t \tag{5.30}$$

Therefore, the velocity of the center of mass at the instant that the paper is completely pulled out is

$${}^{\mathcal{F}}\mathbf{v}_O(\Delta t) = \mu g\Delta t\mathbf{E}_x \tag{5.31}$$

(b) Angular Velocity of Cylinder at Instant When Paper is Removed

Applying Euler's law about the center of mass of the cylinder, we have that

$$\bar{\mathbf{M}} = \frac{{}^{\mathcal{F}}d}{dt}\left({}^{\mathcal{F}}\bar{\mathbf{H}}\right) \Longleftrightarrow \mathbf{M}_O = \frac{{}^{\mathcal{F}}d}{dt}\left({}^{\mathcal{F}}\mathbf{H}_O\right) \tag{5.32}$$

Now since $\mathbf{N}$ and $m\mathbf{g}$ pass through point O, the moment about O is due to only $\mathbf{F}_f$. Consequently,

$$\bar{\mathbf{M}} = (\mathbf{r}_P - \bar{\mathbf{r}}) \times \mathbf{F}_f \tag{5.33}$$

Substituting the earlier expressions for $\mathbf{r}_P - \bar{\mathbf{r}}$ and $\mathbf{F}_f$, we obtain

$$\bar{\mathbf{M}} = r\mathbf{E}_y \times \mu m g \mathbf{E}_x = -r\mu m g \mathbf{E}_z \tag{5.34}$$

Furthermore,

$${}^{\mathcal{F}}\bar{\mathbf{H}} = \bar{\mathbf{I}}^{\mathcal{R}} \cdot {}^{\mathcal{F}}\boldsymbol{\omega}^{\mathcal{R}} \tag{5.35}$$

Now since $\{\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_z\}$ is a principal-axis basis, we have that

$$\bar{\mathbf{I}}^{\mathcal{R}} = I_{rr}\mathbf{e}_r \otimes \mathbf{e}_r + I_{\theta\theta}\mathbf{e}_\theta \otimes \mathbf{e}_\theta + I_{zz}\mathbf{e}_z \otimes \mathbf{e}_z \tag{5.36}$$

Using the angular velocity ${}^{\mathcal{F}}\boldsymbol{\omega}^{\mathcal{R}}$ from Eq. (5.5), we obtain ${}^{\mathcal{F}}\bar{\mathbf{H}}$ as

$${}^{\mathcal{F}}\bar{\mathbf{H}} = \frac{mr^2}{2}\omega\mathbf{e}_z \tag{5.37}$$

Differentiating ${}^{\mathcal{F}}\bar{\mathbf{H}}$ in reference frame $\mathcal{F}$, we have

$$\frac{{}^{\mathcal{F}}d}{dt}\left({}^{\mathcal{F}}\bar{\mathbf{H}}\right) = \frac{mr^2}{2}\dot{\omega}\mathbf{e}_z \tag{5.38}$$

Then, applying Euler's second law, we have

$$-r\mu m g = \frac{mr^2}{2}\dot{\omega} \tag{5.39}$$

Integrating this last equation from 0 to Δt, we obtain

$$\int_0^{\Delta t} -r\mu m g dt = \int_0^{\Delta t} \frac{mr^2}{2}\dot{\omega} dt \tag{5.40}$$

We then obtain

$$-r\mu m g \Delta t = \frac{mr^2}{2}\omega(\Delta t) - \frac{mr^2}{2}\omega(t=0) \tag{5.41}$$

Noting that the disk is initially stationary, we have that

$$\omega(t=0) = 0 \tag{5.42}$$

Therefore,

$$-r\mu m g \Delta t = \frac{mr^2}{2}\omega(\Delta t) \tag{5.43}$$

Solving for $\omega(\Delta t)$, we obtain

$$\omega(\Delta t) = -\frac{2\mu g}{r}\Delta t \tag{5.44}$$

The angular velocity of the disk at the instant that the paper is pulled out is then given as

$${}^{\mathcal{F}}\boldsymbol{\omega}^{\mathcal{R}}(\Delta t) = -\frac{2\mu g}{r}\Delta t\mathbf{e}_z \tag{5.45}$$

Question 5–2

A collar of mass m_1 is attached to a rod of mass m_2 and length l as shown in Fig. P5-2. The collar slides without friction along a horizontal track while the rod is free to rotate about the pivot point Q located at the collar. Knowing that the angle θ describes the orientation of the rod with the vertical, that x is the horizontal position of the cart, and that gravity acts downward, determine a system of two differential equations for the collar and the rod in terms of x and θ.

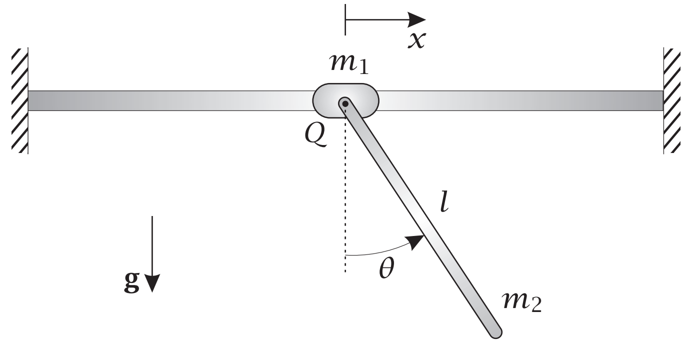

Figure P5-2

Solution to Question 5–2

Preliminaries

For this problem it is convenient to apply the following balance laws:

- Newton's 2^{nd} law to the collar
- Euler's 1^{st} law to the center of mass of the rod
- Euler's 2^{nd} law about the center of mass of the rod

In order to use the aforementioned balance laws, we will need the following kinematic quantities in an inertial reference frame:

- The acceleration of the collar
- The acceleration of the center of mass of the rod
- The rate of change of angular momentum of the rod relative to the center of mass of the rod

Kinematics

First, let $\mathcal{F}$ be a fixed reference frame. Then, choose the following coordinate system fixed in $\mathcal{F}$:

$$
\begin{array}{ccc}
 & \text{Origin at Collar When } x = 0 & \\
\mathbf{E}_x & = & \text{To The Right} \\
\mathbf{E}_z & = & \text{Out of Page} \\
\mathbf{E}_y & = & \mathbf{E}_z \times \mathbf{E}_x
\end{array}
$$

Next, let $\mathcal{R}$ be a reference frame fixed to the rod. Then, choose the following coordinate system fixed in $\mathcal{R}$:

$$
\begin{array}{ccc}
 & \text{Origin at Collar} & \\
\mathbf{e}_r & = & \text{Along Rod} \\
\mathbf{E}_z & = & \text{Out of Page} \\
\mathbf{e}_\theta & = & \mathbf{E}_z \times \mathbf{e}_r
\end{array}
$$

The relationship between the bases $\{\mathbf{E}_x, \mathbf{E}_y, \mathbf{E}_z\}$ and $\{\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_z\}$ is shown in Fig. 5-2 Using Fig. 5-2, we have that

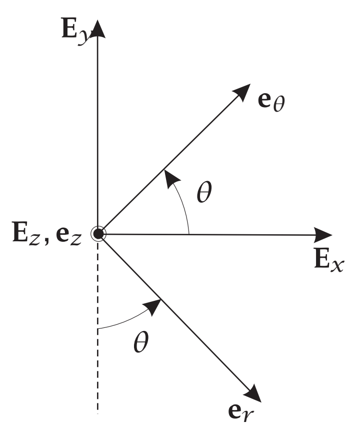

Figure 5-2 Relationship Between Bases $\{\mathbf{E}_x, \mathbf{E}_y, \mathbf{E}_z\}$ and $\{\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_z\}$ for Question 6.2.

$$
\begin{array}{rcl}
\mathbf{e}_r & = & \sin\theta\,\mathbf{E}_x - \cos\theta\,\mathbf{E}_y \\
\mathbf{e}_\theta & = & \cos\theta\,\mathbf{E}_x + \sin\theta\,\mathbf{E}_y \\
\mathbf{E}_x & = & \sin\theta\,\mathbf{e}_r + \cos\theta\,\mathbf{e}_\theta \\
\mathbf{E}_y & = & -\cos\theta\,\mathbf{e}_r + \sin\theta\,\mathbf{e}_\theta
\end{array}
\tag{5.46}
$$

In terms of the basis $\{\mathbf{E}_x, \mathbf{E}_y, \mathbf{E}_z\}$, the position of the collar is given as

$$
\mathbf{r} = x\mathbf{E}_x \tag{5.47}
$$

Therefore, the velocity of the collar in reference frame $\mathcal{F}$ is given as

$$^{\mathcal{F}}\mathbf{v} = \dot{x}\mathbf{E}_x \tag{5.48}$$

Furthermore, the acceleration of the collar in reference frame $\mathcal{F}$ is given as

$$^{\mathcal{F}}\mathbf{a} = \ddot{x}\mathbf{E}_x \tag{5.49}$$

Next, the position of the center of mass of the rod relative to the collar is given as

$$\bar{\mathbf{r}} - \mathbf{r} = \frac{l}{2}\mathbf{e}_r \tag{5.50}$$

In addition, the angular velocity of $\mathcal{R}$ in reference frame $\mathcal{F}$ is given as

$$^{\mathcal{F}}\boldsymbol{\omega}^{\mathcal{R}} = \dot{\theta}\mathbf{E}_z \tag{5.51}$$

Differentiating $^{\mathcal{F}}\boldsymbol{\omega}^{\mathcal{R}}$ in Eq. (5.51), the angular acceleration of reference frame $\mathcal{R}$ in reference frame $\mathcal{F}$ is given as

$$^{\mathcal{F}}\boldsymbol{\alpha}^{\mathcal{R}} = \ddot{\theta}\mathbf{E}_z \tag{5.52}$$

Then, since the location of the collar is also a point on the rod, the acceleration of the center of mass of the rod relative to the collar is obtained from rigid body kinematics as

$$^{\mathcal{F}}\bar{\mathbf{a}} - {}^{\mathcal{F}}\mathbf{a} = {}^{\mathcal{F}}\boldsymbol{\alpha}^{\mathcal{R}} \times (\bar{\mathbf{r}} - \mathbf{r}) + {}^{\mathcal{F}}\boldsymbol{\omega}^{\mathcal{R}} \times \left[{}^{\mathcal{F}}\boldsymbol{\omega}^{\mathcal{R}} \times (\bar{\mathbf{r}} - \mathbf{r})\right] \tag{5.53}$$

Using the expression for $\bar{\mathbf{r}}-\mathbf{r}$ from Eq. (5.50), the expression for $^{\mathcal{F}}\boldsymbol{\omega}^{\mathcal{R}}$ from Eq. (5.51), and the expression for Eq. (5.52), we obtain

$$^{\mathcal{F}}\bar{\mathbf{a}} - {}^{\mathcal{F}}\mathbf{a} = \ddot{\theta}\mathbf{E}_z \times \left(\frac{l}{2}\mathbf{e}_r\right) + \dot{\theta}\mathbf{E}_z \times \left[\dot{\theta}\mathbf{E}_z \times \left(\frac{l}{2}\mathbf{e}_r\right)\right] \tag{5.54}$$

Simplifying Eq. (5.54), we obtain

$$^{\mathcal{F}}\bar{\mathbf{a}} - {}^{\mathcal{F}}\mathbf{a} = -\frac{l}{2}\dot{\theta}^2\mathbf{e}_r + \frac{l}{2}\ddot{\theta}\mathbf{e}_\theta \tag{5.55}$$

Then, using the fact that $^{\mathcal{F}}\mathbf{a} = \ddot{x}\mathbf{E}_x$, we obtain

$$^{\mathcal{F}}\bar{\mathbf{a}} = \ddot{x}\mathbf{E}_x - \frac{l}{2}\dot{\theta}^2\mathbf{e}_r + \frac{l}{2}\ddot{\theta}\mathbf{e}_\theta \tag{5.56}$$

Finally, we need $^{\mathcal{F}}d^{\mathcal{F}}\bar{\mathbf{H}}/dt$. We have that

$$^{\mathcal{F}}\bar{\mathbf{H}} = \bar{\mathbf{I}}^{\mathcal{R}} \cdot {}^{\mathcal{F}}\boldsymbol{\omega}^{\mathcal{R}} \tag{5.57}$$

where $\bar{\mathbf{I}}^{\mathcal{R}}$ is the moment of inertia tensor of the rod relative to the center of mass and ${}^{\mathcal{F}}\boldsymbol{\omega}^{\mathcal{R}}$ is the angular velocity of the rod in reference frame $\mathcal{F}$. Now since the $\{\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_z\}$ is a principal-axis basis, we have that

$$\bar{\mathbf{I}}^{\mathcal{R}} = \bar{I}_{rr}\mathbf{e}_r \otimes \mathbf{e}_r + \bar{I}_{\theta\theta}\mathbf{e}_\theta \otimes \mathbf{e}_\theta + \bar{I}_{zz}\mathbf{E}_z \otimes \mathbf{E}_z \tag{5.58}$$

Furthermore, using the expression for ${}^{\mathcal{F}}\boldsymbol{\omega}^{\mathcal{R}}$ as given in Eq. (5.51), we obtain

$${}^{\mathcal{F}}\bar{\mathbf{H}} = \bar{I}_{zz}\dot{\theta}\mathbf{E}_z \tag{5.59}$$

Now, for a slender rod of mass M and length l we have that

$$\bar{I}_z z = \frac{Ml^2}{12} \tag{5.60}$$

Therefore,

$${}^{\mathcal{F}}\bar{\mathbf{H}} = \frac{Ml^2}{12}\dot{\theta}\mathbf{E}_z \tag{5.61}$$

Differentiating the expression in Eq. (5.61), we obtain

$$\frac{{}^{\mathcal{F}}d}{dt}\left({}^{\mathcal{F}}\bar{\mathbf{H}}\right) = \frac{Ml^2}{12}\ddot{\theta}\mathbf{E}_z \tag{5.62}$$

Kinetics

As stated earlier, to solve this problem we will use the following balance laws:

- Newton's 2^{nd} law to the collar
- Euler's 1^{st} law to the center of mass of the rod
- Euler's 2^{nd} law about the center of mass of the rod

Application of Newton's 2^{nd} Law to Collar

The free body diagram of the collar is shown in Fig. 5-3. where

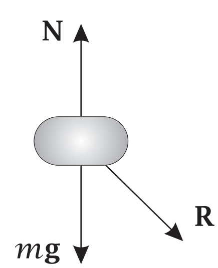

Figure 5-3 Free Body Diagram of Collar for Question 6.2.

$$\begin{array}{rcl} \mathbf{N} & = & \text{Force of Track on Collar} \\ \mathbf{R} & = & \text{Reaction Force of Hinge Due to Rod} \\ m\mathbf{g} & = & \text{Force of Gravity} \end{array}$$

Then, from the geometry we have that[1]

$$\begin{array}{rcl} \mathbf{N} & = & N\mathbf{E}_y \\ \mathbf{R} & = & R_r\mathbf{e}_r + R_\theta\mathbf{e}_\theta \\ m\mathbf{g} & = & -mg\mathbf{E}_y \end{array} \tag{5.63}$$

Using Eq. (5.46), the force $\mathbf{R}$ can be written as

$$\mathbf{R} = R_r(\sin\theta\,\mathbf{E}_x - \cos\theta\,\mathbf{E}_y) + R_\theta(\cos\theta\,\mathbf{E}_x + \sin\theta\,\mathbf{E}_y) \tag{5.64}$$

which gives

$$\mathbf{R} = (R_r\sin\theta + R_\theta\cos\theta)\mathbf{E}_x + (-R_r\cos\theta + R_\theta\sin\theta)\mathbf{E}_y \tag{5.65}$$

The total force on the collar is then given as

$$\begin{aligned} \mathbf{F} &= \mathbf{N} + \mathbf{R} + m\mathbf{g} \\ &= N\mathbf{E}_y + (R_r\sin\theta + R_\theta\cos\theta)\mathbf{E}_x + (-R_r\cos\theta + R_\theta\sin\theta)\mathbf{E}_y - mg\mathbf{E}_y \end{aligned} \tag{5.66}$$

This gives

$$\mathbf{F} = (R_r\sin\theta + R_\theta\cos\theta)\mathbf{E}_x + (N - mg - R_r\cos\theta + R_\theta\sin\theta)\mathbf{E}_y \tag{5.67}$$

Setting $\mathbf{F}$ equal to $m\,{}^{\mathcal{F}}\mathbf{a}$ using the expression for ${}^{\mathcal{F}}\mathbf{a}$ from Eq. (5.49), we obtain

$$(R_r\sin\theta + R_\theta\cos\theta)\mathbf{E}_x + (N - mg - R_r\cos\theta + R_\theta\sin\theta)\mathbf{E}_y = m\ddot{x}\mathbf{E}_x \tag{5.68}$$

Equating components, we obtain the following two scalar equations:

$$m\ddot{x} = R_r\sin\theta + R_\theta\cos\theta \tag{5.69}$$

$$N - mg - R_r\cos\theta + R_\theta\sin\theta = 0 \tag{5.70}$$

Application of Euler's 1^{st} ***Law to Rod***

Using the free body diagram of the rod as shown in Fig. 5-4, we have that

$$\mathbf{F} = -\mathbf{R} + M\mathbf{g} = -R_r\mathbf{e}_r - R_\theta\mathbf{e}_\theta - Mg\mathbf{E}_y \tag{5.71}$$

Also, equating $\mathbf{F}$ and $m\,{}^{\mathcal{F}}\bar{\mathbf{a}}$ using ${}^{\mathcal{F}}\bar{\mathbf{a}}$ from Eq. (5.56), we have that

$$-R_r\mathbf{e}_r - R_\theta\mathbf{e}_\theta - Mg\mathbf{E}_y = M\left(\ddot{x}\mathbf{E}_x + \frac{l}{2}\ddot{\theta}\mathbf{e}_\theta - \frac{l}{2}\dot{\theta}^2\mathbf{e}_r\right) \tag{5.72}$$

[1] It is noted that, because the rod is a *distributed* mass, the reaction force between the collar and the rod is *not* purely along the direction of the rod. Instead, the reaction force between the collar and the rod has a component orthogonal to the rod.

Eq. (5.72) can be split into components in the $\mathbf{e}_r$ and $\mathbf{e}_\theta$ directions by taking dot products. We note that Furthermore, we note that

$$
\begin{array}{rcl}
\mathbf{E}_x \cdot \mathbf{e}_r & = & \sin\theta \\
\mathbf{E}_y \cdot \mathbf{e}_r & = & -\cos\theta \\
\mathbf{E}_x \cdot \mathbf{e}_\theta & = & \cos\theta \\
\mathbf{E}_y \cdot \mathbf{e}_\theta & = & \sin\theta
\end{array}
\tag{5.73}
$$

Taking dot products in the $\mathbf{e}_r$ and $\mathbf{e}_\theta$ directions, respectively, we obtain the following two scalar equations:

$$-R_r + Mg\cos\theta = M\ddot{x}\sin\theta - \frac{Ml}{2}\dot{\theta}^2 \tag{5.74}$$

$$-R_\theta - Mg\sin\theta = M\ddot{x}\cos\theta + \frac{Ml}{2}\ddot{\theta} \tag{5.75}$$

Application of Euler's 2^{nd} ***Law to Rod***

Referring again to the free body diagram of the rod as shown in Fig. 5-4, we have that where

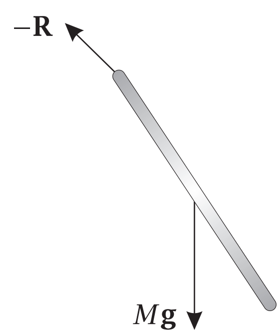

Figure 5-4 Free Body Diagram of Rod for Question 6.4.

$$
\begin{array}{rcl}
-\mathbf{R} & = & \text{Reaction Force of Cart on Rod} \\
M\mathbf{g} & = & \text{Force of Gravity}
\end{array}
$$

Now since gravity passes through the center of mass of the rod, the only moment about the center of mass is due to $-\mathbf{R}$. Consequently,

$$\bar{\mathbf{M}} = (\mathbf{r}_R - \bar{\mathbf{r}}) \times (-\mathbf{R}) \tag{5.76}$$

Also,

$$
\begin{array}{rcl}
\mathbf{r}_R & = & x\mathbf{E}_x \\
\bar{\mathbf{r}} & = & x\mathbf{E}_x + \dfrac{l}{2}\mathbf{e}_r
\end{array}
\tag{5.77}
$$

Consequently,

$$\mathbf{r}_R - \bar{\mathbf{r}} = -\frac{l}{2}\mathbf{e}_r \tag{5.78}$$

Then,

$$\bar{\mathbf{M}} = -\frac{l}{2}\mathbf{e}_r \times (-R_r\mathbf{e}_r - R_\theta\mathbf{e}_\theta) \tag{5.79}$$

This gives

$$\bar{\mathbf{M}} = \frac{l}{2}R_\theta\mathbf{E}_z \tag{5.80}$$

Equating $\bar{\mathbf{M}}$ and ${}^{\mathcal{F}}d^{\mathcal{F}}\bar{\mathbf{H}}/dt$ using the expression for ${}^{\mathcal{F}}d^{\mathcal{F}}\bar{\mathbf{H}}/dt$ from Eq. (5.62), we obtain

$$\frac{Ml^2}{12}\ddot{\theta} = \frac{l}{2}R_\theta \tag{5.81}$$

This gives

$$R_\theta = \frac{Ml}{6}\ddot{\theta} \tag{5.82}$$

System of Two Differential Equations

The system of two differential equations can be obtained from Eq. (5.69), Eq. (5.70), Eq. (5.82), Eq. (5.74), and Eq. (5.75). Substituting Eq. (5.82) into Eq. (5.75), we obtain

$$-Mg\sin\theta - \frac{Ml}{6}\ddot{\theta} = M\ddot{x}\cos\theta + \frac{Ml}{2}\ddot{\theta} \tag{5.83}$$

Simplifying this last equation yields the first differential equation as

$$l\ddot{\theta} + \frac{3}{2}\ddot{x}\cos\theta + \frac{3g}{2}\sin\theta = 0 \tag{5.84}$$

Next, multiplying Eq. (5.74) by $\sin\theta$ and Eq. (5.75) by $\cos\theta$, we have the following two equations:

$$\begin{aligned} -R_r\sin\theta + Mg\cos\theta\sin\theta &= M\ddot{x}\sin^2\theta - \frac{Ml}{2}\dot{\theta}^2\sin\theta \\ -R_\theta\cos\theta - Mg\sin\theta\cos\theta &= M\ddot{x}\cos^2\theta + \frac{Ml}{2}\ddot{\theta}\cos\theta \end{aligned} \tag{5.85}$$

Adding these last two equations gives

$$-R_r\sin\theta - R_\theta\cos\theta = M\ddot{x}(\sin^2\theta + \cos^2\theta) - \frac{Ml}{2}\dot{\theta}^2\sin\theta + \frac{Ml}{2}\ddot{\theta}\cos\theta \tag{5.86}$$

We then obtain

$$-R_r\sin\theta - R_\theta\cos\theta = M\ddot{x} - \frac{Ml}{2}\dot{\theta}^2\sin\theta + \frac{Ml}{2}\ddot{\theta}\cos\theta \tag{5.87}$$

Now substitute the expression for $-R_r \sin\theta - R_\theta \cos\theta$ from Eq. (5.69) into this Eq. (5.87). This gives

$$-m\ddot{x} = M\ddot{x} - \frac{Ml}{2}\dot{\theta}^2 \sin\theta + \frac{Ml}{2}\ddot{\theta}\cos\theta \tag{5.88}$$

Rearranging this last equation, we obtain the second differential equation as

$$(M+m)\ddot{x} - \frac{Ml}{2}\dot{\theta}^2 \sin\theta + \frac{Ml}{2}\ddot{\theta}\cos\theta = 0 \tag{5.89}$$

The system of differential equations is then given as

$$\begin{aligned} l\ddot{\theta} + \frac{3}{2}\ddot{x}\cos\theta + \frac{3g}{2}\sin\theta &= 0 \\ (M+m)\ddot{x} - \frac{Ml}{2}\dot{\theta}^2 \sin\theta + \frac{Ml}{2}\ddot{\theta}\cos\theta &= 0 \end{aligned} \tag{5.90}$$

Question 5–3

A bulldozer pushes a boulder of mass m with a known force $\mathbf{P}$ up a hill inclined at a constant inclination angle β as shown in Fig. P5-3. For simplicity, the boulder is modeled as a uniform sphere of mass m and radius r. Assuming that the boulder rolls without slip along the surface of the hill, that the coefficient of dynamic Coulomb friction between the bulldozer and the boulder is μ, that the force $\mathbf{P}$ is along the direction of the incline and passes through the center of mass of the boulder, and that gravity acts downward, determine the differential equation of motion of the boulder in terms of the variable x.

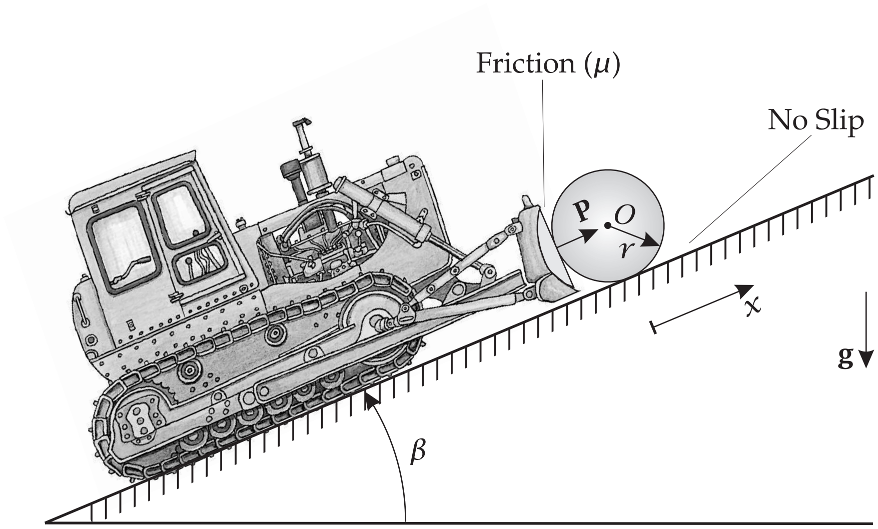

Figure P5-3

Solution to Question 5–3

First, let $\mathcal{F}$ be a fixed reference frame. Then, choose the following coordinate system fixed in $\mathcal{F}$:

$$
\begin{array}{ccc}
& \text{Origin at Point } O \text{ when } x = 0 & \\
\mathbf{E}_x & = & \text{Up Incline} \\
\mathbf{E}_z & = & \text{Into Page} \\
\mathbf{E}_y & = & \mathbf{E}_z \times \mathbf{E}_x
\end{array}
$$

Next, let $\mathcal{B}$ be the boulder. Then, choose the following coordinate system fixed in $\mathcal{B}$:

$$
\begin{array}{ccc}
& \text{Origin at Point } O & \\
\mathbf{e}_r & = & \text{Fixed in } \mathcal{B} \\
\mathbf{e}_z & = & \text{Into Page} \\
\mathbf{e}_\theta & = & \mathbf{e}_z \times \mathbf{e}_r
\end{array}
$$

Now for this problem it is helpful to do some of the kinetics before proceeding with the remainder of the kinematics. First, the free body diagram of the sphere is shown in Fig. 5-5 where P is the point on the sphere that instantaneously is sliding on the plate and Q is the point on the sphere that is instantaneously in contact with the incline.

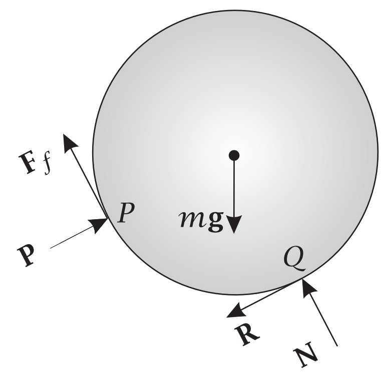

Figure 5-5 Free Body Diagram of Sphere for Question 5–3.

Using Fig. 5-5, the forces acting on the sphere are given as

$\mathbf{N}$	=	Force of Incline on Sphere
$\mathbf{R}$	=	Force of Rolling
$\mathbf{P}$	=	Force of Plate on Sphere
$m\mathbf{g}$	=	Force of Gravity
$\mathbf{F}_f$	=	Force of Friction Due to Contact of Sphere with Bulldozer

From the geometry we have that

$$\mathbf{N} = N\mathbf{E}_y \tag{5.91}$$
$$\mathbf{R} = R\mathbf{E}_x \tag{5.92}$$
$$\mathbf{P} = P\mathbf{E}_x \tag{5.93}$$
$$m\mathbf{g} = mg\mathbf{u}_v \tag{5.94}$$
$$\mathbf{F}_f = -\mu\|\mathbf{P}\|\frac{\mathbf{v}_{\text{rel}}}{\|\mathbf{v}_{\text{rel}}\|} \tag{5.95}$$

where $\mathbf{u}_v$ is the unit vector in the vertically downward direction. Now $\mathbf{u}_v$ is shown in Fig. 5-6.
Using Fig. 5-6, we have that

$$\mathbf{u}_v = -\sin\beta\mathbf{E}_x + \cos\beta\mathbf{E}_y \tag{5.96}$$

Therefore, the force of gravity is obtained as

$$m\mathbf{g} = -mg\sin\beta\mathbf{E}_x + mg\cos\beta\mathbf{E}_y \tag{5.97}$$

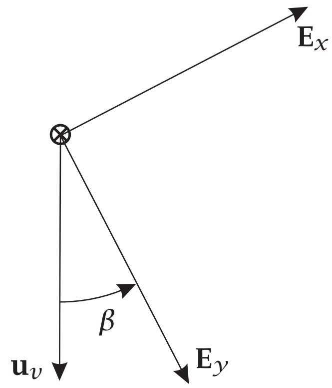

Figure 5-6 Unit Vector in Vertically Downward Direction for Question 5–3.

Next, in order to obtain the correct direction for the friction force $\mathbf{F}_f$, we need to determine $\mathbf{v}_{\text{rel}}$. We note that

$$\mathbf{v}_{\text{rel}} = {}^{\mathcal{F}}\mathbf{v}_P^{\mathcal{R}} - {}^{\mathcal{F}}\mathbf{v}_P^{\text{plate}} \tag{5.98}$$

where ${}^{\mathcal{F}}\mathbf{v}_P^{\mathcal{R}}$ is the velocity of point P on the sphere in reference frame $\mathcal{F}$ and ${}^{\mathcal{F}}\mathbf{v}_P^{\text{plate}}$ is the of point P on the plate velocity of the plate in reference frame $\mathcal{F}$ (we note in this case that the plate is the surface on which the sphere slides). Now we have from the geometry of the problem that

$$\mathbf{r}_P = x\mathbf{E}_x - r\mathbf{E}_x = (x - r)\mathbf{E}_x \tag{5.99}$$

Consequently,

$$ {}^{\mathcal{F}}\mathbf{v}_P^{\text{plate}} = \frac{{}^{\mathcal{F}}d}{dt}(\mathbf{r}_P) = \dot{x}\mathbf{E}_x \tag{5.100}$$

Next, we need to determine ${}^{\mathcal{F}}\mathbf{v}_P^{\mathcal{R}}$. Since the sphere rolls without slip along the incline, we have that

$$ {}^{\mathcal{F}}\mathbf{v}_Q = \mathbf{0} \tag{5.101}$$

Also,

$$ {}^{\mathcal{F}}\mathbf{v}_O = {}^{\mathcal{F}}\mathbf{v}_Q + {}^{\mathcal{F}}\boldsymbol{\omega}^{\mathcal{R}} \times (\mathbf{r}_O - \mathbf{r}_Q) \tag{5.102}$$

where $\mathcal{R}$ is the reference frame of the sphere and ${}^{\mathcal{F}}\boldsymbol{\omega}^{\mathcal{R}}$ is the angular velocity of the sphere in reference frame $\mathcal{F}$. Because for this problem the motion is planar, we have that

$$ {}^{\mathcal{F}}\boldsymbol{\omega}^{\mathcal{R}} = \omega\mathbf{E}_z \tag{5.103}$$

Furthermore, from the definition of the coordinate system above we have that

$$\mathbf{r}_Q = x\mathbf{E}_x + r\mathbf{E}_y \tag{5.104}$$

$$\mathbf{r}_O = x\mathbf{E}_x \tag{5.105}$$

Consequently,

$$\mathbf{r}_O - \mathbf{r}_Q = -r\mathbf{E}_y \tag{5.106}$$

We then have that

$$\mathbf{v}_O = \omega\mathbf{E}_z \times (-r\mathbf{E}_y) = r\omega\mathbf{E}_x \tag{5.107}$$

Furthermore, a second expression for $\mathbf{r}_O$ is given as

$$\mathbf{r}_O = x\mathbf{E}_x \tag{5.108}$$

Therefore,

$${}^{\mathcal{F}}\mathbf{v}_O = \frac{{}^{\mathcal{F}}d}{dt}(\mathbf{r}_O) = \dot{x}\mathbf{E}_x \tag{5.109}$$

Differentiating ${}^{\mathcal{F}}\mathbf{v}_O$ in Eq. (5.109), the acceleration of point O in reference frame $\mathcal{F}$ is given as

$${}^{\mathcal{F}}\mathbf{a}_O = \ddot{x}\mathbf{E}_x \tag{5.110}$$

Setting the result of Eq. (5.107) equal to the result of Eq.(5.109), we obtain

$$\dot{x} = r\omega \tag{5.111}$$

which gives

$$\omega = \frac{\dot{x}}{r} \tag{5.112}$$

Differentiating this last result, we obtain

$$\dot{\omega} = \frac{\ddot{x}}{r} \tag{5.113}$$

The angular velocity of the sphere is then given as

$${}^{\mathcal{F}}\boldsymbol{\omega}^{\mathcal{R}} = \frac{\dot{x}}{r}\mathbf{E}_z \tag{5.114}$$

Furthermore, the angular acceleration of the sphere in reference frame $\mathcal{F}$ is given as

$${}^{\mathcal{F}}\boldsymbol{\alpha}^{\mathcal{R}} = \frac{{}^{\mathcal{F}}d}{dt}\left({}^{\mathcal{F}}\boldsymbol{\alpha}^{\mathcal{R}}\right) = \frac{\ddot{x}}{r}\mathbf{E}_z \tag{5.115}$$

We can then use the expression for ω from Eq. (5.112) to determine ${}^{\mathcal{F}}\mathbf{v}_P$. We have that

$${}^{\mathcal{F}}\mathbf{v}_P = {}^{\mathcal{F}}\mathbf{v}_Q + {}^{\mathcal{F}}\boldsymbol{\omega}^{\mathcal{R}} \times (\mathbf{r}_P - \mathbf{r}_Q) \tag{5.116}$$

Then, using Eq. (5.99) and Eq. (5.104), we have that

$$\mathbf{r}_P - \mathbf{r}_Q = -r\mathbf{E}_x - r\mathbf{E}_y \tag{5.117}$$

Consequently,

$${}^{\mathcal{F}}\mathbf{v}_P^{\mathcal{R}} = \omega\mathbf{E}_z \times (-r\mathbf{E}_x - r\mathbf{E}_y) = \dot{x}Ex - \dot{x}\mathbf{E}_y \tag{5.118}$$

Then $\mathbf{v}_{\text{rel}}$ is obtained from Eq. (5.100) and Eq. (5.118) as

$$\mathbf{v}_{\text{rel}} = {}^{\mathcal{F}}\mathbf{v}_P^{\mathcal{R}} - {}^{\mathcal{F}}\mathbf{v}_P^{\text{plate}} = -\dot{x}\mathbf{E}_y \tag{5.119}$$

The force of friction $\mathbf{F}_f$ is then given as

$$\mathbf{F}_f = -\mu\|\mathbf{P}\|\frac{-\dot{x}\mathbf{E}_y}{\dot{x}} = \mu P\mathbf{E}_y \tag{5.120}$$

We now have expressions for all of the forces acting the sphere and can proceed to solving parts (a) and (b).

Determination of Differential Equation of Motion

The differential equation of motion can be obtained using Euler's law about the point of contact of the sphere with the incline (i.e. about point Q), i.e. we can apply

$$\mathbf{M}_Q - (\bar{\mathbf{r}} - \mathbf{r}_Q) \times m{}^{\mathcal{F}}\mathbf{a}_Q = \frac{{}^{\mathcal{F}}d}{dt}\left({}^{\mathcal{F}}\mathbf{H}_Q\right) \tag{5.121}$$

Noting that $\bar{\mathbf{r}} = \mathbf{r}_O$, we have that

$$\mathbf{M}_Q - (\mathbf{r}_O - \mathbf{r}_Q) \times m{}^{\mathcal{F}}\mathbf{a}_Q = \dot{\mathbf{H}}_Q \tag{5.122}$$

Now the acceleration of the contact point Q is obtained as

$${}^{\mathcal{F}}\mathbf{a}_Q = {}^{\mathcal{F}}\mathbf{a}_O + {}^{\mathcal{F}}\boldsymbol{\alpha}^{\mathcal{R}} \times (\mathbf{r}_Q - \mathbf{r}_O) + {}^{\mathcal{F}}\boldsymbol{\omega}^{\mathcal{R}} \times [\boldsymbol{\omega} \times (\mathbf{r}_Q - \mathbf{r}_O)] \tag{5.123}$$

Using ${}^{\mathcal{F}}\mathbf{a}_O$ from Eq. (5.110), ${}^{\mathcal{F}}\boldsymbol{\omega}^{\mathcal{R}}$ from Eq. (5.114), ${}^{\mathcal{F}}\boldsymbol{\alpha}^{\mathcal{R}}$ from Eq. (5.115), and the fact that $\mathbf{r}_Q - \mathbf{r}_O = r\mathbf{E}_y$, we obtain ${}^{\mathcal{F}}\mathbf{a}_Q$ as

$$\begin{aligned}
{}^{\mathcal{F}}\mathbf{a}_Q &= \ddot{x}\mathbf{E}_x + \frac{\ddot{x}}{r}\mathbf{E}_z \times r\mathbf{E}_y + \frac{\dot{x}}{r}\mathbf{E}_z \times \left[\frac{\dot{x}}{r}\mathbf{E}_z \times r\mathbf{E}_y\right] \\
&= \ddot{x}\mathbf{E}_x - \ddot{x}\mathbf{E}_x - \frac{\dot{x}^2}{r}\mathbf{E}_y \\
&= -\frac{\dot{x}^2}{r}\mathbf{E}_y
\end{aligned} \tag{5.124}$$

Consequently, the inertial moment $-(\mathbf{r}_O - \mathbf{r}_Q) \times m{}^{\mathcal{F}}\mathbf{a}_Q$ is given as

$$-r\mathbf{E}_y \times m(-\frac{\dot{x}^2}{r}\mathbf{E}_y) = \mathbf{0} \tag{5.125}$$

Since the inertial moment is zero, for this problem Eq. (5.121) reduces to

$$\mathbf{M}_Q = \frac{{}^{\mathcal{F}}d}{dt}\left({}^{\mathcal{F}}\mathbf{H}_Q\right) \tag{5.126}$$

Next, looking at the free body diagram above, it can be seen that the forces **N** and **R** both pass through point Q. Therefore, the moment relative to point Q is due to only the forces **P**, $\mathbf{F}_f$, and $m\mathbf{g}$. We then have that

$$\mathbf{M}_Q = (\mathbf{r}_P - \mathbf{r}_Q) \times \mathbf{P} + (\mathbf{r}_P - \mathbf{r}_Q) \times \mathbf{F}_f + (\mathbf{r}_O - \mathbf{r}_Q) \times m\mathbf{g} \tag{5.127}$$

where

$$\mathbf{r}_P - \mathbf{r}_Q = -r\mathbf{E}_x - r\mathbf{E}_y \tag{5.128}$$

$$\mathbf{r}_O - \mathbf{r}_Q = -r\mathbf{E}_y \tag{5.129}$$

Therefore,

$$\begin{aligned}\mathbf{M}_Q &= (-r\mathbf{E}_x - r\mathbf{E}_y) \times F\mathbf{E}_x + (-r\mathbf{E}_x - r\mathbf{E}_y) \times \mu P\mathbf{E}_y \\ &+ (-r\mathbf{E}_y) \times (-mg\sin\beta\mathbf{E}_x + mg\cos\beta\mathbf{E}_y)\end{aligned} \tag{5.130}$$

This last expression simplifies to

$$\mathbf{M}_Q = rP\mathbf{E}_z - r\mu P\mathbf{E}_z = rP(1-\mu)\mathbf{E}_z - mgr\sin\beta\mathbf{E}_z \tag{5.131}$$

Furthermore, the angular momentum relative to the contact point is given as

$$^{\mathcal{F}}\mathbf{H}_Q = {}^{\mathcal{F}}\bar{\mathbf{H}} + (\mathbf{r}_Q - \mathbf{r}_O) \times m({}^{\mathcal{F}}\mathbf{v}_Q - {}^{\mathcal{F}}\mathbf{v}_O) \tag{5.132}$$

Now we have that

$$^{\mathcal{F}}\bar{\mathbf{H}} = \bar{\mathbf{I}}^{\mathcal{R}} \cdot {}^{\mathcal{F}}\boldsymbol{\omega}^{\mathcal{R}} \tag{5.133}$$

Now since $\{\mathbf{E}_x, \mathbf{E}_y, \mathbf{E}_z\}$ is a principal-axis basis, we have that

$$\bar{\mathbf{I}}^{\mathcal{R}} = \bar{I}_{rr}\mathbf{e}_r \otimes \mathbf{e}_r + \bar{I}_{\theta\theta}\mathbf{e}_\theta \otimes \mathbf{e}_\theta + \bar{I}_{zz}\mathbf{e}_z \otimes \mathbf{e}_z \tag{5.134}$$

Then, substituting $\bar{\mathbf{I}}^{\mathcal{R}}$ from Eq. (5.134) into Eq. (5.133), we obtain

$$^{\mathcal{F}}\bar{\mathbf{H}} = \bar{I}_{zz}\omega\mathbf{e}_z = \bar{I}_{zz}\frac{\dot{x}}{r}\mathbf{e}_z \tag{5.135}$$

Now we have for a uniform sphere we have that $\bar{I}_{zz} = 2mr^2/5$. Consequently,

$$^{\mathcal{F}}\bar{\mathbf{H}} = \frac{2mr^2}{5}\frac{\dot{x}}{r}\mathbf{e}_z = \frac{2mr\dot{x}}{5}\mathbf{e}_z \tag{5.136}$$

Next, since ${}^{\mathcal{F}}\mathbf{v}_Q = \mathbf{0}$, we have that

$$(\mathbf{r}_Q - \mathbf{r}_O) \times m({}^{\mathcal{F}}\mathbf{v}_Q - {}^{\mathcal{F}}\mathbf{v}_O) = mr\mathbf{E}_y \times (-\dot{x}\mathbf{E}_x) = mr\dot{x}\mathbf{E}_z \tag{5.137}$$

Consequently,

$$^{\mathcal{F}}\mathbf{H}_Q = \frac{2mr\dot{x}}{5}\mathbf{E}_z + mr\dot{x}\mathbf{e}_z = \frac{7mr\dot{x}}{5}\mathbf{e}_z \tag{5.138}$$

Differentiating ${}^{\mathcal{F}}\mathbf{H}_Q$ in reference frame $\mathcal{F}$, we obtain

$$\frac{{}^{\mathcal{F}}d}{dt}\left({}^{\mathcal{F}}\mathbf{H}_Q\right) = \frac{7mr\ddot{x}}{5}\mathbf{e}_z \tag{5.139}$$

Equating $\mathbf{M}_Q$ from Eq. (5.131) with ${}^{\mathcal{F}}d\left({}^{\mathcal{F}}\mathbf{H}_Q\right)/dt$ from Eq. (5.139), we obtain

$$rP(1-\mu) - mgr\sin\beta = \frac{7}{5}mr\ddot{x} \tag{5.140}$$

Simplifying Eq. (5.140), we obtain the differential equation of motion as

$$\frac{7}{5}m\ddot{x} + mg\sin\beta = P(1-\mu) \tag{5.141}$$

Question 5–4

A uniform slender rod of mass m and length l pivots about its center at the fixed point O as shown in Fig. P5-4. A torsional spring with spring constant K is attached to the rod at the pivot point. The rod is initially at rest and the spring is uncoiled when a linear impulse $\hat{\mathbf{F}}$ is applied transversely at the lower end of the rod. Determine (a) the angular velocity of the rod immediately after the impulse $\hat{\mathbf{F}}$ is applied and (b) the maximum angle $\theta_{\max}$ attained by the rod after the impulse is applied.

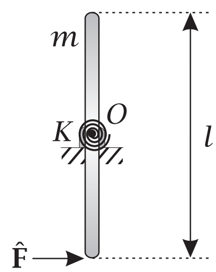

Figure P5-4

Solution to Question 5–4

Preliminaries

We note for this problem that the fixed point O is the center of mass. Furthermore, since the rod is constrained to rotate about point O, we need not consider the translational motion of the center of mass of the rod. Then, since the only motion that needs to be considered is the rotational motion of the rod about its center of mass, the only kinematic quantity of interest in this problem is the angular momentum of the rod.

Kinematics

First, let $\mathcal{F}$ be a fixed reference frame. Then, choose the following coordinate system fixed in reference frame $\mathcal{F}$:

$$\begin{array}{lcl} & \text{Origin at Point } O & \\ \mathbf{E}_x & = & \text{Along Rod (Down)} \\ \mathbf{E}_z & = & \text{Out of Page} \\ \mathbf{E}_y & = & \mathbf{E}_z \times \mathbf{E}_x \end{array}$$

Next, let $\mathcal{R}$ be a reference frame fixed to the rod. Then, choose the following coordinate system fixed in reference frame $\mathcal{R}$:

Origin at Point O		
$\mathbf{e}_r$	=	Along Rod (Down)
$\mathbf{e}_z$	=	$\mathbf{E}_z$
$\mathbf{e}_\theta = \mathbf{e}_z \times \mathbf{e}_r$		

Then, since the rod rotates about the $\mathbf{e}_z$-direction, the angular velocity of the rod in reference frame $\mathcal{F}$ can then be expressed as

$$^{\mathcal{F}}\boldsymbol{\omega}^{\mathcal{R}} = \omega \mathbf{e}_z \tag{5.142}$$

Furthermore, the angular momentum of the rod in reference frame $\mathcal{F}$ relative to the point O is given as

$$^{\mathcal{F}}\mathbf{H}_O = \mathbf{I}_O^{\mathcal{R}} \cdot {}^{\mathcal{F}}\boldsymbol{\omega}^{\mathcal{R}} \tag{5.143}$$

Now since $\{\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_z\}$ is a principal-axis basis, the inertia tensor can be expressed as

$$\mathbf{I}_O^{\mathcal{R}} = I_{rr}^O \mathbf{e}_r \otimes \mathbf{e}_r + I_{\theta\theta}^O \mathbf{e}_\theta \otimes \mathbf{e}_\theta + I_{zz}^O \mathbf{e}_z \otimes \mathbf{e}_z \tag{5.144}$$

Substituting the results of Eq. (5.144) and Eq. (5.142) into Eq. (5.143), we obtain $^{\mathcal{F}}\mathbf{H}_O$ as

$$^{\mathcal{F}}\mathbf{H}_O = \left(I_{rr}^O \mathbf{e}_r \otimes \mathbf{e}_r + I_{\theta\theta}^O \mathbf{e}_\theta \otimes \mathbf{e}_\theta + I_{zz}^O \mathbf{e}_z \otimes \mathbf{e}_z \right) \cdot \omega \mathbf{e}_z = I_{zz}^O \omega \mathbf{e}_z \tag{5.145}$$

Finally, we have for a slender uniform rod that

$$I_{zz}^O = \frac{ml^2}{12} \tag{5.146}$$

Substituting the result of Eq. (5.146) into Eq. (5.145), we obtain

$$^{\mathcal{F}}\mathbf{H}_O = \frac{ml^2}{12} \omega \mathbf{e}_z \tag{5.147}$$

Kinetics

Since the center of mass of the rod is the fixed point O, in order to solve this problem we only need to apply angular impulse and angular momentum relative to the center of mass. From this point forward we will use the general notation for the center of mass rather than using the subscript O. Then, applying angular impulse and angular momentum relative to point O, we have that

$$\hat{\bar{\mathbf{M}}} = {}^{\mathcal{F}}\bar{\mathbf{H}}' - {}^{\mathcal{F}}\bar{\mathbf{H}} \tag{5.148}$$

The free body diagram of the rod during the application of $\hat{\mathbf{F}}$ is shown in Fig. 5-7. Now, we first note that the impulse due to the torsional spring, $\boldsymbol{\tau}_s$, is *zero* because $\hat{\mathbf{F}}$

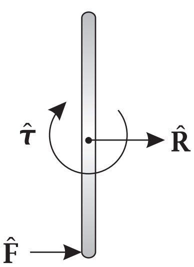

Figure 5-7 Free Body Diagram of Rod During Application of $\hat{\mathbf{F}}$ for Problem 5–4.

is assumed to be applied instantaneously and, thus, the orientation of the rod does not change during the application of $\hat{\mathbf{F}}$. Next, we see that the reaction impulse at point O, $\hat{\mathbf{R}}$, is inconsequential because $\hat{\mathbf{R}}$ passes through point O. Therefore, the impulse applied to the rod about point O is given as

$$\hat{\bar{\mathbf{M}}} = (\mathbf{r}_{\hat{F}} - \bar{\mathbf{r}}) \times \hat{\mathbf{F}} \tag{5.149}$$

Now we see that

$$\mathbf{r}_{\hat{F}} = \frac{l}{2}\mathbf{e}_r \tag{5.150}$$

Furthermore, since $\hat{\mathbf{F}}$ is applied horizontally, we have that

$$\hat{\mathbf{F}} = \hat{F}\mathbf{e}_\theta \tag{5.151}$$

Substituting Eq. (5.150) and Eq. (5.151) into Eq. (5.149), we obtain

$$\hat{\bar{\mathbf{M}}} = \frac{l}{2}\mathbf{e}_r \times \hat{F}\mathbf{e}_\theta = \frac{l\hat{F}}{2}\mathbf{E}_z \tag{5.152}$$

Next, since the rod is initially at rest, we have that

$$^{\mathcal{F}}\bar{\mathbf{H}} = \mathbf{0} \tag{5.153}$$

Then, substituting ω_2 into the expression for $\mathbf{H}_O$ from Eq. (5.143), we have that

$$^{\mathcal{F}}\bar{\mathbf{H}}' = \frac{ml^2}{12}\omega'\mathbf{E}_z \tag{5.154}$$

Setting $\hat{\mathbf{M}}$ from Eq. (5.152) equal to $^{\mathcal{F}}\bar{\mathbf{H}}'$ from Eq. (5.154, we obtain

$$\frac{l\hat{F}}{2}\mathbf{E}_z = \frac{ml^2}{12}\omega'\mathbf{E}_z \tag{5.155}$$

Dropping $\mathbf{E}_z$ and solving for ω', we obtain

$$\omega' = \frac{6\hat{F}}{ml} \tag{5.156}$$

The angular velocity of the rod the instant after the impulse $\hat{\mathbf{F}}$ is applied is then given as

$$\left({}^{\mathcal{F}}\boldsymbol{\omega}^{\mathcal{R}}\right)' = \frac{6\hat{F}}{ml}\mathbf{e}_z \tag{5.157}$$

Now, after $\hat{\mathbf{F}}$ has been applied, the rod starts to rotate. Therefore, the only forces and torques acting on the rod after the application of $\hat{\mathbf{F}}$ are the reaction force, $\mathbf{R}$, at point O and the spring torque, $\boldsymbol{\tau}_s$. Since ${}^{\mathcal{F}}\bar{\mathbf{v}} = \mathbf{0}$, we see that $\mathbf{R}\cdot\mathbf{v}_O = 0$ which implies that $\mathbf{R}$ does no work. Furthermore, the spring torque $\boldsymbol{\tau}_s$ is conservative with potential energy

$${}^{\mathcal{F}}U = {}^{\mathcal{F}}U_s = \frac{1}{2}K\theta^2 \tag{5.158}$$

Since the only forces or torques acting on the rod after $\hat{\mathbf{F}}$ is applied are conservative or do no work, energy is conserved. The total energy of the rod is then given as

$${}^{\mathcal{F}}E = {}^{\mathcal{F}}T + {}^{\mathcal{F}}U \tag{5.159}$$

The kinetic energy is given as

$${}^{\mathcal{F}}T = \frac{1}{2}m\,{}^{\mathcal{F}}\bar{\mathbf{v}}\cdot{}^{\mathcal{F}}\bar{\mathbf{v}} + \frac{1}{2}{}^{\mathcal{F}}\bar{\mathbf{H}}\cdot{}^{\mathcal{F}}\boldsymbol{\omega}^{\mathcal{R}} \tag{5.160}$$

since ${}^{\mathcal{F}}\bar{\mathbf{v}} = \mathbf{0}$, the kinetic energy reduces to

$$T = \frac{1}{2}{}^{\mathcal{F}}\bar{\mathbf{H}}\cdot{}^{\mathcal{F}}\boldsymbol{\omega}^{\mathcal{R}} \tag{5.161}$$

Substituting $\frac{1}{2}{}^{\mathcal{F}}\bar{\mathbf{H}}$ from Eq. (5.143) into Eq. (5.161), we obtain

$${}^{\mathcal{F}}T = \frac{ml^2\omega^2}{24} \tag{5.162}$$

Observing that $\omega = \dot{\theta}$, we have that

$${}^{\mathcal{F}}T = \frac{ml^2\dot{\theta}^2}{24} \tag{5.163}$$

Now the point where θ attains its maximum value is where $\dot{\theta} = 0$. Applying conservation of energy, we have that

$${}^{\mathcal{F}}T_1 + {}^{\mathcal{F}}U_1 = {}^{\mathcal{F}}T_2 + {}^{\mathcal{F}}U_2 \tag{5.164}$$

where point "1" is immediately after $\hat{\mathbf{F}}$ is applied and point "2" is when $\dot{\theta} = 0$. We then have that

$${}^{\mathcal{F}}T_1 = \frac{ml^2\omega_1^2}{24} \tag{5.165}$$

Now we note that $\omega_1 = \omega'$ from Eq. (5.156). Consequently, we obtain ${}^{\mathcal{F}}T_1$ as

$$ {}^{\mathcal{F}}T_1 = \frac{ml^2}{24}\left[\frac{6\hat{F}}{ml}\right]^2 \tag{5.166} $$

Furthermore, we have that $U_1 = 0$ since the spring is initially uncoiled. Next, we see that $T_2 = 0$ since $\omega_2 = \dot{\theta}_2 = 0$, i.e.

$$ {}^{\mathcal{F}}T_2 = \frac{ml^2\omega_2^2}{24} = \frac{ml^2\dot{\theta}_2^2}{24} = 0 \tag{5.167} $$

Last, we have that

$$ {}^{\mathcal{F}}U_2 = \frac{1}{2}K\theta_{\text{max}}^2 \tag{5.168} $$

Then, applying Eq. (5.164), we obtain

$$ \frac{ml^2}{24}\left[\frac{6\hat{F}}{ml}\right]^2 = \frac{1}{2}K\theta_{\text{max}}^2 \tag{5.169} $$

Solving for θ_{max} gives

$$ \theta_{\text{max}} = \frac{6\hat{F}}{ml}\sqrt{\frac{ml^2}{12K}} \tag{5.170} $$

Question 5–5

A homogeneous cylinder of mass m and radius r moves along a surface inclined at a constant inclination angle β as shown in Fig. P5-5. The surface of the incline is composed of a frictionless segment of known length x between points A and B and a segment with a coefficient of friction μ from point B onwards. Knowing that the cylinder is released from rest at point A and that gravity acts vertically downward, determine (a) the velocity of the center of mass and the angular velocity when the disk reaches point B, (b) the time (measured from point B) when sliding stops and rolling begins, and (c) the velocity of the center of mass and the angular velocity of the disk when sliding stops and rolling begins.

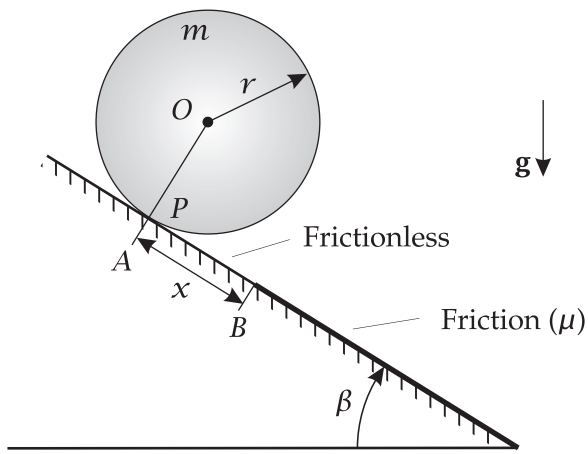

Figure P5-5

Solution to Question 5–5

Kinematics

Let $\mathcal{F}$ be the ground. Then, choose the following coordinate system fixed in reference frame $\mathcal{F}$:

$$\begin{array}{ccr} & \text{Origin at } O & \\ & \text{at} t = 0 & \\ \mathbf{E}_x & = & \text{Down Incline} \\ \mathbf{E}_z & = & \text{Into Page} \\ \mathbf{E}_y & = & \mathbf{E}_z \times \mathbf{E}_x \end{array}$$

Next, let $\mathcal{D}$ be the cylinder. Then, choose the following coordinate system fixed in reference frame $\mathcal{D}$:

$$\begin{array}{ccc} & \text{Origin at } O & \\ & \text{at} t = 0 & \\ \mathbf{e}_r & = & \text{Fixed in } \mathcal{D} \\ \mathbf{e}_z & = & \mathbf{E}_z \\ \mathbf{e}_\theta & = & \mathbf{e}_z \times \mathbf{e}_r \end{array}$$

In terms of the basis $\{\mathbf{E}_x, \mathbf{E}_y, \mathbf{E}_z\}$, the position of the center of mass of the cylinder is given as

$$\bar{\mathbf{r}} = \mathbf{r}_O = x\mathbf{E}_x \tag{5.171}$$

Now, since the basis $\{\mathbf{E}_x, \mathbf{E}_y, \mathbf{E}_z\}$ is fixed, the velocity of the center of mass of the cylinder is given as

$${}^{\mathcal{F}}\bar{\mathbf{v}} = \dot{x}\mathbf{E}_x = \bar{v}\mathbf{E}_x \tag{5.172}$$

Finally, the acceleration of the center of mass of the cylinder is given as

$${}^{\mathcal{F}}\bar{\mathbf{a}} = \frac{{}^{\mathcal{F}}d}{dt}\left({}^{\mathcal{F}}\bar{\mathbf{v}}\right) = \ddot{x}\mathbf{E}_x = \dot{\bar{v}} = \bar{a}\mathbf{E}_x \tag{5.173}$$

Next, since the cylinder rotates about the $\mathbf{E}_z$-direction, the angular velocity of the cylinder in the fixed reference frame $\mathcal{F}$ is given as

$${}^{\mathcal{F}}\boldsymbol{\omega}^{\mathcal{R}} = \omega\mathbf{E}_z \tag{5.174}$$

Finally, the velocity of the instantaneous point of contact, P, in reference frame $\mathcal{F}$ is given as

$${}^{\mathcal{F}}\mathbf{v}_P = {}^{\mathcal{F}}\bar{\mathbf{v}} + {}^{\mathcal{F}}\boldsymbol{\omega}^{\mathcal{R}} \times (\mathbf{r}_P - \bar{\mathbf{r}}) = \bar{v}\mathbf{E}_x + \omega\mathbf{E}_z \times r\mathbf{E}_y = (\bar{v} - r\omega)\mathbf{E}_x \tag{5.175}$$

where we note that

$$\mathbf{r}_P - \bar{\mathbf{r}} = r\mathbf{E}_y \tag{5.176}$$

Kinetics

The kinetics of this problem are divided into the following two distinct segments:

(a) the frictionless segment

(b) the segment with friction

We will analyze each of these segments separately.

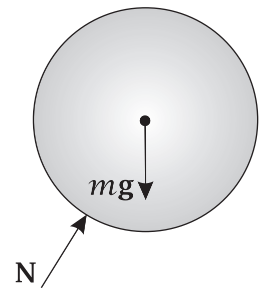

Figure 5-8 Free Body Diagram of Cylinder During Frictionless Segment for Question 5–5.

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Kinetics During Frictionless Segment

The free body diagram of the cylinder during the frictionless segment is shown in Fig. 5-8. It can be seen from Fig. 5-8 that the following forces act on the cylinder during the frictionless segment:

$$
\begin{aligned}
\mathbf{N} &= \text{Normal Force of Incline on Cylinder} \\
m\mathbf{g} &= \text{Force of Gravity}
\end{aligned}
$$

Now from the geometry of the problem we have that

$$\mathbf{N} = N\mathbf{E}_y \tag{5.177}$$

$$m\mathbf{g} = mg\mathbf{u}_v \tag{5.178}$$

$$\tag{5.179}$$

where $\mathbf{u}_v$ is the unit vector in the vertically downward direction as shown in Fig. 5–9.

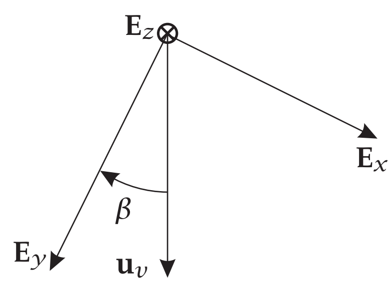

Figure 5-9 Direction of Unit Vertical $\mathbf{u}_v$ in Terms of $\mathbf{E}_x$ and $\mathbf{E}_y$ for Question 5–5.

Using Fig. 5–9, we have

$$\mathbf{u}_v = \sin\beta\mathbf{E}_x + \cos\beta\mathbf{E}_y \tag{5.180}$$

Consequently, the force of gravity is given as

$$m\mathbf{g} = mg\sin\beta\mathbf{E}_x + mg\cos\beta\mathbf{E}_y \tag{5.181}$$

Now we know that the force of gravity is conservative. Furthermore, because the normal force **N** acts at point P, we have that

$$\mathbf{N} \cdot {}^{\mathcal{F}}\mathbf{v}_P = N\mathbf{E}_y \cdot (\bar{v} - r\omega)\mathbf{E}_x = 0 \tag{5.182}$$

Eq. (5.182) implies that the power of **N** is zero which implies that **N** does no work. Then, since gravity is the only force other than **N** and is conservative, energy is conserved during the frictionless segment, i.e.,

$$^{\mathcal{F}}E = {}^{\mathcal{F}}T + {}^{\mathcal{F}}U = \text{constant} \tag{5.183}$$

Now the kinetic energy in reference frame $\mathcal{F}$ is given as

$$^{\mathcal{F}}T = \frac{1}{2}m{}^{\mathcal{F}}\bar{\mathbf{v}} \cdot {}^{\mathcal{F}}\bar{\mathbf{v}} + \frac{1}{2}m{}^{\mathcal{F}}\bar{\mathbf{H}} \cdot {}^{\mathcal{F}}\boldsymbol{\omega}^{\mathcal{R}} \tag{5.184}$$

Now we are given that the disk is released from rest which implies that

$$^{\mathcal{F}}\bar{\mathbf{H}}(t = 0) = \mathbf{0} \tag{5.185}$$

Furthermore, from the free body diagram of Fig. 5-8 is is seen that both of the forces that act on the cylinder during the frictionless segment pass through the center of mass of the cylinder. Consequently, the resultant moment about the center of mass of the cylinder during the frictionless segment is zero, i.e.,

$$\bar{\mathbf{M}} = \mathbf{0} \tag{5.186}$$

Then, since $\bar{\mathbf{M}} \equiv \mathbf{0}$ during the frictionless segment, we have that

$$\frac{^{\mathcal{F}}d}{dt}\left({}^{\mathcal{F}}\bar{\mathbf{H}}\right) = \mathbf{0} \tag{5.187}$$

Eq. (5.187) implies that

$$^{\mathcal{F}}\bar{\mathbf{H}} = \text{constant} \tag{5.188}$$

during the frictionless segment. Then, Eq. (5.185), together with Eq. (5.187) implies that

$$^{\mathcal{F}}\bar{\mathbf{H}} = \mathbf{0} \tag{5.189}$$

during the frictionless segment. Consequently, the kinetic energy of the cylinder during the frictionless segment reduces to

$$^{\mathcal{F}}T = \frac{1}{2}m{}^{\mathcal{F}}\bar{\mathbf{v}} \cdot {}^{\mathcal{F}}\bar{\mathbf{v}} \tag{5.190}$$

Substituting the expression for ${}^{\mathcal{F}}\bar{\mathbf{v}}$ from Eq. (5.172) into Eq. (5.190), we obtain the kinetic energy as

$$^{\mathcal{F}}T = \frac{1}{2}m\bar{v}\mathbf{E}_x \cdot \bar{v}\mathbf{E}_x = \frac{1}{2}m\bar{v}^2 \tag{5.191}$$

Next, since the only conservative force acting on the cylinder is that due to gravity, the potential energy in reference frame $\mathcal{F}$ is given as

$$^{\mathcal{F}}U = {}^{\mathcal{F}}U_g = -m\mathbf{g} \cdot \bar{\mathbf{r}} \tag{5.192}$$

Substituting the results of Eq. (5.181) and Eq. (5.171) into Eq. (5.192), we obtain the potential energy in reference frame $\mathcal{F}$ as

$$^{\mathcal{F}}U = {}^{\mathcal{F}}U_g = -(mg\sin\beta\mathbf{E}_x + mg\cos\beta\mathbf{E}_y) \cdot x\mathbf{E}_x = -mgx\sin\beta \tag{5.193}$$

Then, substituting the results of Eq. (5.191) and Eq. (5.193) into Eq. (5.183), we obtain the total energy of the cylinder during the frictionless segment as

$$^{\mathcal{F}}E = \frac{1}{2}m\bar{v}^2 - mgx\sin\beta = \text{constant} \tag{5.194}$$

Then, using the principle of work and energy for a rigid body, we have that

$$^{\mathcal{F}}E_0 = {}^{\mathcal{F}}E_1 \tag{5.195}$$

where $^{\mathcal{F}}E_0$ and $^{\mathcal{F}}E_1$ are the total energies of the cylinder at the beginning and end of the frictionless segment. Eq. (5.195) implies that

$$\frac{1}{2}m\bar{v}_0^2 - mgx_0\sin\beta = \frac{1}{2}m\bar{v}_1^2 - mgx_1\sin\beta \tag{5.196}$$

Now we know that $\bar{v}_0 = 0$ and x_0 are both zero. Therefore,

$$\frac{1}{2}m\bar{v}_1^2 - mgx_1\sin\beta = 0 \tag{5.197}$$

Then, knowing that $x_1 = x$, we can solve Eq. (5.197) for $\bar{v}_1$ to give

$$\bar{v}_1 = \sqrt{2gx\sin\beta} \tag{5.198}$$

Eq. (5.198) implies that the velocity of the center of mass of the cylinder at the end of the frictionless segment is given as

$$^{\mathcal{F}}\bar{\mathbf{v}}(t_1) = \sqrt{2gx\sin\beta}\mathbf{E}_x \tag{5.199}$$

Finally, since $^{\mathcal{F}}\bar{\mathbf{H}} \equiv \mathbf{0}$ during the frictionless segment, the angular velocity of the cylinder during the frictionless segment is also zero which implies that

$$^{\mathcal{F}}\boldsymbol{\omega}^{\mathcal{R}}(t_1) = \mathbf{0} \tag{5.200}$$

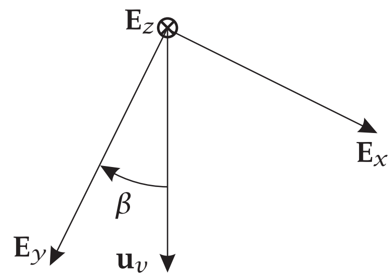

Figure 5-10 Free Body Diagram of Cylinder During Segment with Friction for Question 5–5.

Segment with Friction

The free body diagram of the cylinder during the segment with friction is shown in Fig. 5-10.
It can be seen that the key difference between the segment with friction and the frictionless segment is that a friction force, $\mathbf{F}_f$, acts at the instantaneous point of contact. Recalling that the expression for the force of sliding Coulomb friction is given as

$$\mathbf{F}_f = -\mu\|\mathbf{N}\|\frac{\mathbf{v}_{\text{rel}}}{\|\mathbf{v}_{\text{rel}}\|} \tag{5.201}$$

Now we know that

$$\mathbf{v}_{\text{rel}} = {}^{\mathcal{F}}\mathbf{v}_P{}^{\mathcal{R}} - {}^{\mathcal{F}}\mathbf{v}_P{}^{S} \tag{5.202}$$

where S denotes the inclined surface. Now since the incline is fixed, we have that

$${}^{\mathcal{F}}\mathbf{v}_P{}^{S} = \mathbf{0} \tag{5.203}$$

which implies that

$$\mathbf{v}_{\text{rel}} = {}^{\mathcal{F}}\mathbf{v}_P{}^{\mathcal{R}} \tag{5.204}$$

Then, using the result of Eq. (5.175), we have that

$$\mathbf{v}_{\text{rel}} = (\bar{v} - r\omega)\mathbf{E}_x \tag{5.205}$$

Now we know that, at the *beginning* of the segment with friction that

$$\bar{v}(t_1) - r\omega(t_1) = \bar{v}(t_1) > 0 \tag{5.206}$$

Therefore, during the period when the cylinder is sliding, we must have that

$$\bar{v} - r\omega > 0 \tag{5.207}$$

which implies that

$$|\bar{v} - r\omega| = \bar{v} - r\omega \tag{5.208}$$

Therefore, we have that

$$\frac{\mathbf{v}_{\text{rel}}}{\|\mathbf{v}_{\text{rel}}\|} = \frac{(\bar{v} - r\omega)\mathbf{E}_x}{|\bar{v} - r\omega|} = \mathbf{E}_x \tag{5.209}$$

The force of sliding Coulomb friction is then given as

$$\mathbf{F}_f = -\mu\|\mathbf{N}\|\mathbf{E}_x \tag{5.210}$$

Now, in order to solve for the velocity of the center of mass of the cylinder and the angular velocity of the cylinder at the instant that sliding stops and rolling begins, we need to apply both the principle of linear impulse and linear momentum and the principle of angular impulse and angular momentum. First, we can apply the principle of linear impulse and linear momentum by applying Euler's 1^{st} law to the cylinder during the segment with friction as

$$\mathbf{F} = {}^{\mathcal{F}}\bar{\mathbf{a}} \tag{5.211}$$

First, the resultant force acting on the cylinder during the segment with friction is given as

$$\mathbf{F} = \mathbf{N} + m\mathbf{g} + \mathbf{F}_f \tag{5.212}$$

Using the expressions for $\mathbf{N}$, $m\mathbf{g}$, and $\mathbf{F}_f$ from Eq. (5.177), Eq. (5.181), and Eq. (5.210), we obtain the resultant force acting on the cylinder as

$$\mathbf{F} = N\mathbf{E}_y + mg\sin\beta\mathbf{E}_x + mg\cos\beta\mathbf{E}_y - \mu\|\mathbf{N}\|\mathbf{E}_x \tag{5.213}$$

Then, using $\mathbf{F}$ from Eq. (5.213) and the expression for ${}^{\mathcal{F}}\bar{\mathbf{a}}$ from Eq. (5.173) in Eq. (5.211), we have that

$$N\mathbf{E}_y + mg\sin\beta\mathbf{E}_x + mg\cos\beta\mathbf{E}_y - \mu\|\mathbf{N}\|\mathbf{E}_x = m\ddot{x}\mathbf{E}_x \tag{5.214}$$

Simplifying Eq. (5.214), we obtain

$$(mg\sin\beta - \mu\|\mathbf{N}\|)\mathbf{E}_x + (N + mg\cos\beta)\mathbf{E}_y - \mu\|\mathbf{N}\|\mathbf{E}_x = m\ddot{x}\mathbf{E}_x \tag{5.215}$$

Equating components in Eq. (5.215) yields the following two scalar equations:

$$mg\sin\beta - \mu\|\mathbf{N}\| = m\ddot{x} \tag{5.216}$$

$$N + mg\cos\beta = 0 \tag{5.217}$$

Eq. (5.217) implies that

$$N = -mg\cos\beta \tag{5.218}$$

Consequently, the magnitude of the normal force, $\|\mathbf{N}\|$, is given as

$$\|\mathbf{N}\| = mg\cos\beta \tag{5.219}$$

Substituting $\|\mathbf{N}\|$ from Eq. (5.219) into Eq. (5.216) gives

$$mg\sin\beta - \mu mg\cos\beta = m\ddot{x} \tag{5.220}$$

Eq. (5.220) simplifies to

$$g(\sin\beta - \mu\cos\beta) = \ddot{x} \tag{5.221}$$

Integrating Eq. (5.221) from $t = t_1$ to $t = t_2$ where t_2 is the time when sliding stops and rolling begins, we have that

$$\int_{t_1}^{t_2} g(\sin\beta - \mu\cos\beta)dt = \int_{t_1}^{t_2} \ddot{x}dt = \dot{x}(t_2) - \dot{x}(t_1) = \bar{v}(t_2) - \bar{v}(t_1) \tag{5.222}$$

Now since the quantity $g(\sin\beta - \mu\cos\beta)$ is constant, we have that

$$\int_{t_1}^{t_2} g(\sin\beta - \mu\cos\beta)dt = g(\sin\beta - \mu\cos\beta)(t_2 - t_1) \tag{5.223}$$

Substituting the result of Eq. (5.223) into Eq. (5.222) gives

$$g(\sin\beta - \mu\cos\beta)(t_2 - t_1) = \bar{v}(t_2) - \bar{v}(t_1) \tag{5.224}$$

Next, we can apply the principle of angular impulse and angular momentum to the cylinder during the segment with friction indirectly by applying Euler's 2^{nd} law. Using the center of mass of the cylinder as the reference point, we have Euler's 2^{nd} law as

$$\bar{\mathbf{M}} = \frac{^{\mathcal{F}}d}{dt}\left(^{\mathcal{F}}\bar{\mathbf{H}}\right) \tag{5.225}$$

Now since $\mathbf{N}$ and $m\mathbf{g}$ pass through the center of mass, the only moment acting on the cylinder during the friction segment is that due to friction and is given as

$$\bar{\mathbf{M}} = (\mathbf{r}_P - \bar{\mathbf{r}}) \times \mathbf{F}_f \tag{5.226}$$

Recalling that $\mathbf{r}_P - \bar{\mathbf{r}} = r\mathbf{E}_y$ and using the expression for $\mathbf{F}_f$ from Eq. (5.210), we have that

$$\bar{\mathbf{M}} = r\mathbf{E}_y \times (-\mu\|\mathbf{N}\|\mathbf{E}_x) = r\mathbf{E}_y \times (-\mu mg\cos\beta\mathbf{E}_x) = r\mu mg\cos\beta\mathbf{E}_z \tag{5.227}$$

Furthermore, the angular momentum of the cylinder relative to the center of mass in reference frame $\mathcal{F}$ is given as

$$^{\mathcal{F}}\bar{\mathbf{H}} = \bar{\mathbf{I}}^{\mathcal{R}} \cdot {^{\mathcal{F}}}\boldsymbol{\omega}^{\mathcal{R}} \tag{5.228}$$

Since $\{\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_z\}$ is a principal-axis basis, we can write the moment of inertia tensor of the cylinder as

$$\bar{\mathbf{I}}^{\mathcal{R}} = \bar{I}_{rr}\mathbf{e}_r \otimes \mathbf{e}_r + \bar{I}_{\theta\theta}\mathbf{e}_\theta \otimes \mathbf{e}_\theta + \bar{I}_{zz}\mathbf{e}_z \otimes \mathbf{e}_z \tag{5.229}$$

Then, using the expression for ${}^{\mathcal{F}}\boldsymbol{\omega}^{\mathcal{R}}$ from Eq. (5.174), we have that

$$ {}^{\mathcal{F}}\bar{\mathbf{H}} = \bar{I}_{zz}\omega\mathbf{e}_z = \frac{mr^2}{2} \tag{5.230} $$

where $\bar{I}_{zz} = mr^2/2$ for a uniform cylinder. Therefore,

$$ {}^{\mathcal{F}}\bar{\mathbf{H}} = \frac{mr^2}{2}\omega\mathbf{e}_z \tag{5.231} $$

Integrating Eq. (5.225) from t_1 to t_2, we obtain

$$ \int_{t_1}^{t_2} \bar{\mathbf{M}} dt = \int_{t_1}^{t_2} \frac{{}^{\mathcal{F}}d}{dt}\left({}^{\mathcal{F}}\bar{\mathbf{H}}\right) dt = {}^{\mathcal{F}}\bar{\mathbf{H}}(t_2) - {}^{\mathcal{F}}\bar{\mathbf{H}}(t_1) \tag{5.232} $$

Noting that $r\mu mg\cos\beta$ and the vector $\mathbf{E}_z$ are constant, we have from Eq. (5.227) that

$$ \int_{t_1}^{t_2} \bar{\mathbf{M}} dt = \int_{t_1}^{t_2} r\mu mg\cos\beta\mathbf{E}_z dt = r\mu mg\cos\beta(t_2 - t_1)\mathbf{E}_z \tag{5.233} $$

Substituting the result of Eq. (5.233) and the result of Eq. (5.231) into Eq. (5.232), we obtain

$$ r\mu mg\cos\beta(t_2 - t_1)\mathbf{E}_z = \frac{mr^2}{2}\omega(t_2)\mathbf{E}_z - \frac{mr^2}{2}\omega(t_1)\mathbf{E}_z \tag{5.234} $$

Now we recall the cylinder is *not* rotating at the beginning of the friction segment. Consequently, $\omega(t_1) = 0$. Using this last fact, dropping the dependence on $\mathbf{E}_z$ and solving Eq. (5.234) for $\omega(t_2)$, we obtain

$$ \omega(t_2) = \frac{2g\mu(t_2 - t_1)\cos\beta}{r} \tag{5.235} $$

Lastly, we know that, at time t_2, when sliding stops and rolling begins, we have the following kinematic constraint:

$$ {}^{\mathcal{F}}\mathbf{v}_P(t_2) = \mathbf{0} \tag{5.236} $$

Using the expression for ${}^{\mathcal{F}}\mathbf{v}_P$ from Eq. (5.175), we have that

$$ \bar{v}(t_2) - r\omega(t_2) = 0 \tag{5.237} $$

Solving Eq. (5.237) for $\omega(t_2)$, we obtain

$$ \omega(t_2) = \frac{\bar{v}(t_2)}{r} \tag{5.238} $$

Using Eq. (5.224), Eq. (5.235), and Eq. (5.238), we can solve for the velocity of the center of mass of the cylinder and the angular velocity of the cylinder at time

t_2 when sliding stops and rolling begins. First, substituting the result of Eq. (5.238) into Eq. (5.235), we have that

$$\frac{\bar{v}(t_2)}{r} = \frac{2g\mu(t_2 - t_1)\cos\beta}{r} \tag{5.239}$$

Solving Eq. (5.239) for $\bar{v}(t_2)$, we obtain

$$\bar{v}(t_2) = 2g\mu(t_2 - t_1)\cos\beta \tag{5.240}$$

Then, substituting the result of Eq. (5.240) into Eq. (5.224), we have that

$$g(\sin\beta - \mu\cos\beta)(t_2 - t_1) = 2g\mu(t_2 - t_1)\cos\beta - \bar{v}(t_1) \tag{5.241}$$

Solving Eq. (5.241) for $t_2 - t_1$ gives

$$t_2 - t_1 = \frac{\bar{v}(t_1)}{g(3\mu\cos\beta - \sin\beta)} \tag{5.242}$$

Substituting the result of Eq. (5.242) into Eq. (5.240), we have that

$$\bar{v}(t_2) = \frac{2\mu\cos\beta\bar{v}(t_1)}{(3\mu\cos\beta - \sin\beta)} \tag{5.243}$$

Now we have from the end of the frictionless segment that $\bar{v}(t_1) = \sqrt{2gx\sin\beta}$. Consequently, we have from Eq. (5.243) that

$$\bar{v}(t_2) = \frac{2\mu\cos\beta\sqrt{2gx\sin\beta}}{(3\mu\cos\beta - \sin\beta)} \tag{5.244}$$

The velocity of the center of mass of the cylinder at the instant when sliding stops and rolling begins is then given as

$${}^{\mathcal{F}}\bar{\mathbf{v}}(t_2) = \frac{2\mu\cos\beta\sqrt{2gx\sin\beta}}{(3\mu\cos\beta - \sin\beta)}\mathbf{E}_x \tag{5.245}$$

Finally, substituting the result of Eq. (5.244) into Eq. (5.238), we have that

$$\omega(t_2) = \frac{2\mu\cos\beta\sqrt{2gx\sin\beta}}{r(3\mu\cos\beta - \sin\beta)} \tag{5.246}$$

The angular velocity of the cylinder at the instant when sliding stops and rolling begins is then given as

$${}^{\mathcal{F}}\boldsymbol{\omega}^{\mathcal{R}}(t_2) = \frac{2\mu\cos\beta\sqrt{2gx\sin\beta}}{r(3\mu\cos\beta - \sin\beta)}\mathbf{E}_z \tag{5.247}$$

Question 5–6

One end of a uniform slender rod of mass m and length l slides along a frictionless vertical surface while the other end of the rod slides along a frictionless horizontal surface as shown in Fig. P5-6. The angle θ formed by the rod is measured from the vertical. Knowing that gravity acts vertically downward, determine (a) the differential equation of motion for the rod while it maintains contact with both surfaces and (b) the value of the angle θ at which the rod loses contact with the vertical surface. In obtaining your answers, you may assume that the initial conditions are $\theta(t=0)=0$ and $\dot{\theta}(t=0)=0$.

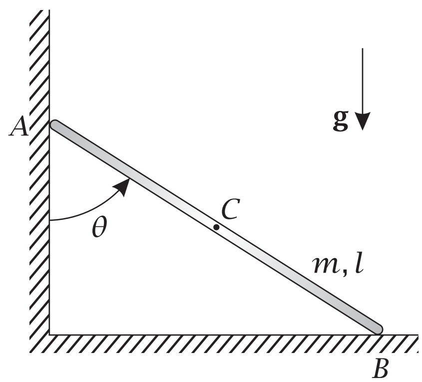

Figure P5-6

Solution to Question 5–6

Kinematics:

First, let $\mathcal{F}$ be the ground. Then, choose the following coordinate system fixed in reference frame $\mathcal{F}$:

$$
\begin{array}{ccr}
 & \text{Origin at Corner} & \\
\mathbf{E}_x & = & \text{To The Right} \\
\mathbf{E}_z & = & \text{Out of Page} \\
\mathbf{E}_y & = & \mathbf{E}_z \times \mathbf{E}_x
\end{array}
$$

Next, let $\mathcal{R}$ be the rod. Then, choose the following coordinate system fixed in reference frame $\mathcal{R}$:

$$
\begin{array}{ccr}
 & \text{Origin at } C & \\
\mathbf{e}_r & = & \text{To The Right} \\
\mathbf{e}_z & = & \mathbf{E}_z \\
\mathbf{e}_\theta & = & \mathbf{e}_z \times \mathbf{e}_r
\end{array}
$$

The position of the center of mass is then given as

$$\bar{\mathbf{r}} = \mathbf{r}_{C/O} = \frac{l}{2}\sin\theta\,\mathbf{E}_x + \frac{l}{2}\cos\theta\,\mathbf{E}_y \tag{5.248}$$

The velocity of the center of mass as viewed by an observer in the inertial reference frame $\mathcal{F}$ is then given as

$${}^{\mathcal{F}}\bar{\mathbf{v}} = \frac{{}^{\mathcal{F}}d}{dt}(\bar{\mathbf{r}}) = \frac{l}{2}\dot{\theta}\cos\theta\,\mathbf{E}_x - \frac{l}{2}\dot{\theta}\sin\theta\,\mathbf{E}_y \tag{5.249}$$

Next, the acceleration of the center of mass as viewed by an observer in the inertial reference frame $\mathcal{F}$ is then given as

$${}^{\mathcal{F}}\bar{\mathbf{a}} = \frac{{}^{\mathcal{F}}d}{dt}\left({}^{\mathcal{F}}\bar{\mathbf{v}}\right) = \frac{l}{2}\left[\ddot{\theta}\cos\theta - \dot{\theta}^2\sin\theta\right]\mathbf{E}_x - \frac{l}{2}\left[\ddot{\theta}\sin\theta + \dot{\theta}^2\cos\theta\right]\mathbf{E}_y \tag{5.250}$$

Also, the angular velocity of the rod in reference frame $\mathcal{F}$ is given as

$${}^{\mathcal{F}}\boldsymbol{\omega}^{\mathcal{R}} = \dot{\theta}\mathbf{E}_z \tag{5.251}$$

which implies that the angular acceleration of the rod is

$${}^{\mathcal{F}}\boldsymbol{\alpha}^{\mathcal{R}} = \ddot{\theta}\mathbf{E}_z \tag{5.252}$$

Kinetics

For this problem it is most convenient to apply Euler's laws using the center of mass of the rod as the reference point. The free body diagram of the rod is given in Fig. 5-11 where

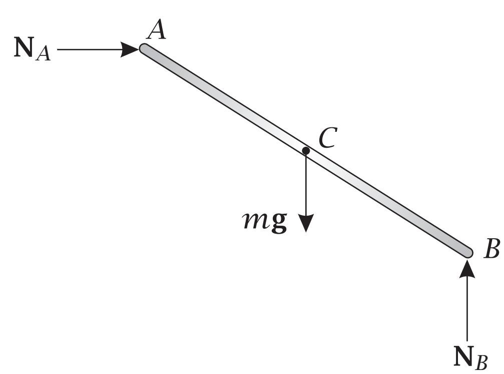

Figure 5-11 Free Body Diagram of Rod for Question 5.7.

$\mathbf{N}_A$	=	Reaction Force of Vertical Wall on Rod
$\mathbf{N}_B$	=	Reaction Force of Floor on Rod
$m\mathbf{g}$	=	Force of Gravity

From the geometry we have that

$$\begin{aligned}\mathbf{N}_A &= N_A\mathbf{E}_x \\ \mathbf{N}_B &= N_B\mathbf{E}_y \\ m\mathbf{g} &= -mg\mathbf{E}_y\end{aligned} \tag{5.253}$$

Therefore, the resultant force on the rod is given as

$$\mathbf{F} = \mathbf{N}_A + \mathbf{N}_B + m\mathbf{g} = N_A\mathbf{E}_x + N_B\mathbf{E}_y - mg\mathbf{E}_y = N_A\mathbf{E}_x + (N_B - mg)\mathbf{E}_y \tag{5.254}$$

Then, applying Euler's 1^{st} law by setting $\mathbf{F}$ equal to $m{}^{\mathcal{F}}\mathbf{a}_C$ where ${}^{\mathcal{F}}\mathbf{a}_C$ is obtained from Eq. (5.250), we have that

$$N_A\mathbf{E}_x + (N_B - mg)\mathbf{E}_y = \frac{ml}{2}\left[\ddot{\theta}\cos\theta - \dot{\theta}^2\sin\theta\right]\mathbf{E}_x - \frac{ml}{2}\left[\ddot{\theta}\sin\theta + \dot{\theta}^2\cos\theta\right]\mathbf{E}_y \tag{5.255}$$

Equating components in Eq. (5.255), we obtain the following two scalar equations:

$$N_A = \frac{ml}{2}\left[\ddot{\theta}\cos\theta - \dot{\theta}^2\sin\theta\right] \tag{5.256}$$

$$N_B - mg = -\frac{ml}{2}\left[\ddot{\theta}\sin\theta + \dot{\theta}^2\cos\theta\right] \tag{5.257}$$

Next, we apply Euler's 2^{nd} law relative to the center of mass. Since the force of gravity passes through the center of mass, the resultant moment about the center of mass is due to only $\mathbf{N}_x$ and $\mathbf{N}_y$ and is given as

$$\bar{\mathbf{M}} = (\mathbf{r}_A - \bar{\mathbf{r}}) \times \mathbf{N}_A + (\mathbf{r}_B - \bar{\mathbf{r}}) \times \mathbf{N}_B \tag{5.258}$$

We note that

$$\begin{aligned}\mathbf{r}_A &= l\cos\theta\,\mathbf{E}_y \\ \mathbf{r}_B &= l\sin\theta\,\mathbf{E}_x\end{aligned} \tag{5.259}$$

Consequently,

$$\begin{aligned}\mathbf{r}_A - \bar{\mathbf{r}} &= l\cos\theta\,\mathbf{E}_y - \left[\frac{l}{2}\sin\theta\,\mathbf{E}_x + \frac{l}{2}\cos\theta\,\mathbf{E}_y\right] = -\frac{l}{2}\sin\theta\,\mathbf{E}_x + \frac{l}{2}\cos\theta\,\mathbf{E}_y \\ \mathbf{r}_B - \bar{\mathbf{r}} &= l\sin\theta\,\mathbf{E}_x - \left[\frac{l}{2}\sin\theta\,\mathbf{E}_x + \frac{l}{2}\cos\theta\,\mathbf{E}_y\right] = \frac{l}{2}\sin\theta\,\mathbf{E}_x - \frac{l}{2}\cos\theta\,\mathbf{E}_y\end{aligned} \tag{5.260}$$

The moment relative to the center of mass of the rod is then given as

$$\begin{aligned}\bar{\mathbf{M}} &= \left[-\frac{l}{2}\sin\theta\,\mathbf{E}_x + \frac{l}{2}\cos\theta\,\mathbf{E}_y\right] \times N_A\mathbf{E}_x + \left[\frac{l}{2}\sin\theta\,\mathbf{E}_x - \frac{l}{2}\cos\theta\,\mathbf{E}_y\right] \times N_B\mathbf{E}_y \\ &= -\frac{l}{2}N_A\cos\theta\,\mathbf{E}_z + \frac{l}{2}N_B\sin\theta\,\mathbf{E}_z = \frac{l}{2}\left(N_B\sin\theta - N_A\cos\theta\right)\mathbf{E}_z\end{aligned} \tag{5.261}$$

Next, we have

$$^{\mathcal{F}}\bar{\mathbf{H}} = \bar{\mathbf{I}}^{\mathcal{R}} \cdot {}^{\mathcal{F}}\boldsymbol{\omega}^{\mathcal{R}} \tag{5.262}$$

Now because $\{\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_z\}$ is a principal-axis basis, we have that

$$\bar{\mathbf{I}}^{\mathcal{R}} = \bar{I}_{rr}\mathbf{e}_r \otimes \mathbf{e}_r + \bar{I}_{\theta\theta}\mathbf{e}_\theta \otimes \mathbf{e}_\theta + \bar{I}_{zz}\mathbf{e}_z \otimes \mathbf{e}_z \tag{5.263}$$

Therefore,

$$^{\mathcal{F}}\bar{\mathbf{H}} = \bar{I}_{zz}{}^{\mathcal{F}}\boldsymbol{\omega}^{\mathcal{R}} = \bar{I}_{zz}\dot{\theta}\mathbf{e}_z \tag{5.264}$$

Now, we know that

$$\bar{I}_{zz} = \frac{ml^2}{12} \tag{5.265}$$

Consequently,

$$^{\mathcal{F}}\bar{\mathbf{H}} = \frac{ml^2}{12}\dot{\theta}\mathbf{e}_z \tag{5.266}$$

The rate of change of $^{\mathcal{F}}\bar{\mathbf{H}}$ as viewed by an observer in the inertial reference frame $\mathcal{F}$ is then given as frame $\mathcal{F}$, we obtain

$$\frac{^{\mathcal{F}}d}{dt}\left({}^{\mathcal{F}}\bar{\mathbf{H}}\right) = \bar{I}_{zz}\ddot{\theta}\mathbf{e}_z \tag{5.267}$$

Applying Euler's second law relative to the center of mass of the rod, we obtain

$$\frac{l}{2}\left(N_B \sin\theta - N_A \cos\theta\right) = \frac{ml^2\ddot{\theta}}{12} \tag{5.268}$$

Simplifying this last result gives

$$N_B \sin\theta - N_A \cos\theta = \frac{ml\ddot{\theta}}{6} \tag{5.269}$$

(a) Differential Equation of Motion During Contact With Wall and Floor

The differential equation of motion can now be determined using the the results of Eqs. (5.256), (5.257), (5.269). First, solving Eq. (5.257) for N_B gives

$$N_B = mg - \frac{ml}{2}\left(\ddot{\theta}\sin\theta + \dot{\theta}^2\cos\theta\right) \tag{5.270}$$

Substituting N_A from Eq. (5.256) and the last expression for N_B into Eq. (5.269), we obtain

$$\left[mg - \frac{ml}{2}(\ddot{\theta}\sin\theta + \dot{\theta}^2\cos\theta)\right]\sin\theta - \frac{ml}{2}\left(\ddot{\theta}\cos\theta - \dot{\theta}^2\sin\theta\right)\cos\theta = \frac{ml}{6}\ddot{\theta} \tag{5.271}$$

This last equation can be rewritten as

$$mg\sin\theta - \frac{ml}{2}\ddot{\theta}\sin^2\theta - \frac{ml}{2}\dot{\theta}^2\cos\theta\sin\theta - \frac{ml}{2}\ddot{\theta}\cos^2\theta + \frac{ml}{2}\dot{\theta}^2\sin\theta\cos\theta = \frac{ml}{6}\ddot{\theta} \tag{5.272}$$

which simplifies to

$$mg\sin\theta - \frac{ml}{2}\ddot{\theta} = \frac{ml}{6}\ddot{\theta}. \tag{5.273}$$

The differential equation of motion while the rod maintains contact with the wall and floor is then given as

$$\ddot{\theta} = \frac{3g}{2l}\sin\theta \tag{5.274}$$

(b) Value of θ When Rod Loses Contact With Vertical Wall

The rod will lose contact with the vertical wall when $\mathbf{N}_A = \mathbf{0}$. Consequently, we need to determine the value of θ such that

$$N_A = \frac{ml}{2}\left[\ddot{\theta}\cos\theta - \dot{\theta}^2\sin\theta\right] = 0 \tag{5.275}$$

It is seen from this last expression that we need to find expressions for $\ddot{\theta}$ and $\dot{\theta}^2$ in terms of θ. We have an expression for $\ddot{\theta}$ in terms of θ from the result of part (a), namely,

$$\ddot{\theta} = \frac{3g}{2l}\sin\theta \tag{5.276}$$

Now we note that

$$\ddot{\theta} = \frac{d\dot{\theta}}{dt} = \frac{d\dot{\theta}}{d\theta}\frac{d\theta}{dt} = \dot{\theta}\frac{d\dot{\theta}}{d\theta} \tag{5.277}$$

Therefore,

$$\dot{\theta}\frac{d\dot{\theta}}{d\theta} = \frac{3g}{2l}\sin\theta \tag{5.278}$$

Separating variables in this last expression, we obtain

$$\dot{\theta}d\dot{\theta} = \frac{3g}{2l}\sin\theta\, d\theta \tag{5.279}$$

Integrating both sides of this last equation gives

$$\int_{\dot{\theta}_0}^{\dot{\theta}} \dot{\theta}d\dot{\theta} = \int_{\theta_0}^{\theta} \frac{3g}{2l}\sin\theta\, d\theta \tag{5.280}$$

which gives

$$\left[\frac{\dot{\theta}^2}{2}\right]_{\dot{\theta}_0}^{\dot{\theta}} = \left[-\frac{3g}{2l}\cos\theta\right]_{\theta_0}^{\theta} \tag{5.281}$$

Noting that $\theta(t=0)=0$ and $\dot{\theta}(t=0)=0$, we obtain

$$\frac{\dot{\theta}^2}{2}=\frac{3g}{2l}(1-\cos\theta) \tag{5.282}$$

Consequently,

$$\dot{\theta}^2=\frac{3g}{l}(1-\cos\theta) \tag{5.283}$$

Substituting this last result for $\dot{\theta}^2$ and the original differential equation from Eq. (5.274) into Eq. (5.275), we obtain

$$\frac{ml}{2}\left[\frac{3g}{2l}\sin\theta\cos\theta-\frac{3g}{l}(1-\cos\theta)\sin\theta\right]=0 \tag{5.284}$$

Simplifying this last expression, we obtain

$$\frac{1}{2}\sin\theta\cos\theta-(1-\cos\theta)\sin\theta=0 \tag{5.285}$$

This gives

$$\sin\theta\,[\cos\theta-2(1-\cos\theta)]=0 \tag{5.286}$$

which simplifies to

$$\sin\theta\,[3\cos\theta-2]=0 \tag{5.287}$$

Therefore, we have that

$$\sin\theta=0 \quad \text{or} \quad 3\cos\theta-2=0 \tag{5.288}$$

This implies that

$$\theta=0 \quad \text{or} \quad \theta=\cos^{-1}(2/3) \tag{5.289}$$

Since $\theta=0$ occurs before the motion starts, we reject this solution. Then the angle θ at which the rod loses contact with the vertical wall is

$$\theta=\cos^{-1}(2/3) \tag{5.290}$$

Question 5–7

A homogeneous semi-circular cylinder of mass m and radius r rolls without slip along a horizontal surface as shown in Fig. P5-7. The center of mass of the cylinder is located at point C while point O is located at the center of the main diameter of the cylinder. Knowing that the angle θ is measured from the vertical and that gravity acts downward, determine the differential equation of motion for the cylinder. In obtaining your answers, you may assume that $4r/(3\pi) \approx 0.42r$.

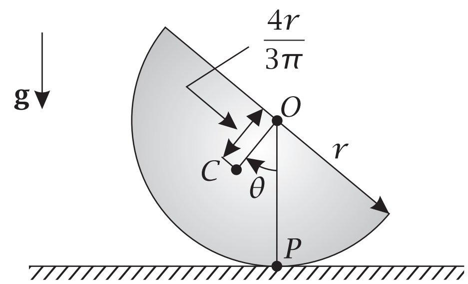

Figure P5-7

Solution to Question 5–7

Kinematics

First, let $\mathcal{F}$ be a fixed reference frame. Then choose the following coordinate system fixed in $\mathcal{F}$:

$$
\begin{array}{lcl}
 & \text{Origin at } O \text{ When } \theta = 0 & \\
\mathbf{E}_x & = & \text{To The Right} \\
\mathbf{E}_z & = & \text{Into Page} \\
\mathbf{E}_y & = & \mathbf{E}_z \times \mathbf{E}_x
\end{array}
$$

Next, let $\mathcal{R}$ be a reference frame fixed to the cylinder. Then choose the following coordinate system fixed in $\mathcal{R}$:

$$
\begin{array}{lcl}
 & \text{Origin at } O \text{ Moving with Cylinder} & \\
\mathbf{e}_r & = & \text{Along } OC \\
\mathbf{e}_z & = & \text{Into Page} (= \mathbf{E}_z) \\
\mathbf{e}_\theta & = & \mathbf{E}_z \times \mathbf{e}_r
\end{array}
$$

The relationship between the bases $\{\mathbf{E}_x, \mathbf{E}_y, \mathbf{E}_z\}$ and $\{\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_z\}$ is shown in Fig. 5-12. Using Fig. 5-12, we have that

$$
\begin{array}{rcl}
\mathbf{e}_r & = & -\sin\theta\,\mathbf{E}_x + \cos\theta\,\mathbf{E}_y \\
\mathbf{e}_\theta & = & -\cos\theta\,\mathbf{E}_x - \sin\theta\,\mathbf{E}_y
\end{array}
\tag{5.291}
$$

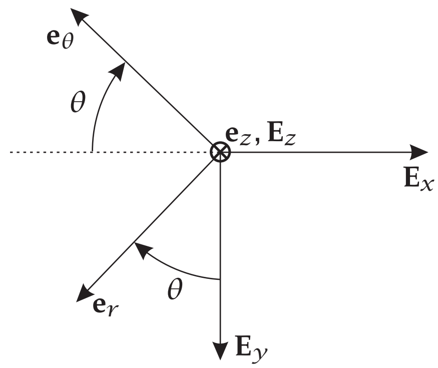

Figure 5-12 Geometry of Coordinate Systems for Question 5–7.

Now since θ is the angle measured from the vertically downward direction, the angular velocity of the cylinder in the fixed reference frame is given as

$$^{\mathcal{F}}\boldsymbol{\omega}^{\mathcal{R}} = \dot{\theta}\mathbf{E}_z \tag{5.292}$$

Next, since the cylinder rolls without slip along a fixed surface, we have that

$$^{\mathcal{F}}\mathbf{v}_P = \mathbf{0} \tag{5.293}$$

Then, we have from kinematics of rigid bodies that

$$^{\mathcal{F}}\mathbf{v}_O - {}^{\mathcal{F}}\mathbf{v}_P = {}^{\mathcal{F}}\mathbf{v}_O = {}^{\mathcal{F}}\boldsymbol{\omega}^{\mathcal{R}} \times (\mathbf{r}_O - \mathbf{r}_P) \tag{5.294}$$

Now, from the geometry of the problem we have that

$$\mathbf{r}_O - \mathbf{r}_P = -r\mathbf{E}_y \tag{5.295}$$

Consequently, we obtain the velocity of point O in reference frame $\mathcal{F}$ as

$$^{\mathcal{F}}\mathbf{v}_O = \dot{\theta}\mathbf{E}_z \times (-r\mathbf{E}_y) = r\dot{\theta}\mathbf{E}_x \tag{5.296}$$

Furthermore, from kinematics of rigid bodies we have that

$$^{\mathcal{F}}\mathbf{v}_C - {}^{\mathcal{F}}\mathbf{v}_O = {}^{\mathcal{F}}\boldsymbol{\omega}^{\mathcal{R}} \times (\mathbf{r}_C - \mathbf{r}_O) \tag{5.297}$$

Now, the position of point O is given as

$$\mathbf{r}_O = x\mathbf{E}_x \tag{5.298}$$

Furthermore, the position of the center of mass of the cylinder is given as

$$\mathbf{r}_C = x\mathbf{E}_x + 0.42r\mathbf{e}_r \tag{5.299}$$

Consequently, we have the position of C relative to O as

$$\mathbf{r}_C - \mathbf{r}_O = 0.42r\mathbf{e}_r \tag{5.300}$$

Therefore,

$$^{\mathcal{F}}\mathbf{v}_C - {}^{\mathcal{F}}\mathbf{v}_O = \dot{\theta}\mathbf{E}_z \times 0.42r\mathbf{e}_r = 0.42r\dot{\theta}\mathbf{e}_\theta \tag{5.301}$$

Then,

$$^{\mathcal{F}}\mathbf{v}_C = {}^{\mathcal{F}}\mathbf{v}_O + 0.42r\dot{\theta}\mathbf{e}_\theta = r\dot{\theta}\mathbf{E}_x + 0.42r\dot{\theta}\mathbf{e}_\theta \tag{5.302}$$

Kinetics

The free body diagram of the cylinder is shown in Fig. 5-13. The forces acting on

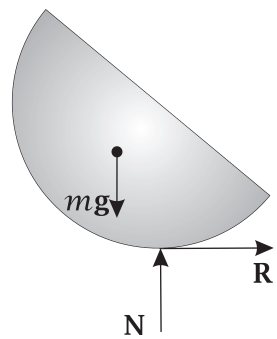

Figure 5-13 Free Body Diagram for Question 5–7.

the cylinder are

$$\begin{aligned} \mathbf{N} &= \text{Reaction Force of Surface on Disk} \\ \mathbf{R} &= \text{Rolling Force} \\ m\mathbf{g} &= \text{Force of Gravity} \end{aligned}$$

Now it is important to notice that the forces **N** and **R** act point P and ${}^{\mathcal{F}}\mathbf{v}_P = \mathbf{0}$. Consequently, neither **N** nor **R** do any work. Since the only other force acting on the cylinder is the conservative force of gravity, we have that

$$ {}^{\mathcal{F}}E = \text{constant} \tag{5.303}$$

Consequently,

$$\frac{d}{dt}\left({}^{\mathcal{F}}E\right) = 0 \tag{5.304}$$

Now we know that

$$ {}^{\mathcal{F}}E = {}^{\mathcal{F}}T + {}^{\mathcal{F}}U \tag{5.305}$$

Since point C is the center of mass of the cylinder, the kinetic energy is given as

$$ {}^{\mathcal{F}}T = \frac{1}{2}m{}^{\mathcal{F}}\mathbf{v}_C \cdot {}^{\mathcal{F}}\mathbf{v}_C + \frac{1}{2}{}^{\mathcal{F}}\mathbf{H}_C \cdot {}^{\mathcal{F}}\boldsymbol{\omega}^{\mathcal{R}} \tag{5.306}$$

Using the expression for ${}^{\mathcal{F}}\mathbf{v}_C$ from Eq. (5.302), we have that

$$\frac{1}{2}m{}^{\mathcal{F}}\mathbf{v}_C \cdot {}^{\mathcal{F}}\mathbf{v}_C = \frac{1}{2}m\left[r\dot{\theta}\mathbf{E}_x + 0.42r\dot{\theta}\mathbf{e}_\theta\right] \cdot \left[r\dot{\theta}\mathbf{E}_x + 0.42r\dot{\theta}\mathbf{e}_\theta\right] \tag{5.307}$$

Simplifying this expression, we obtain

$$\frac{1}{2}m{}^{\mathcal{F}}\mathbf{v}_C \cdot {}^{\mathcal{F}}\mathbf{v}_C = \frac{1}{2}m\left[r^2\dot{\theta}^2 + (0.42r\dot{\theta})^2 + 0.84r^2\dot{\theta}^2\mathbf{E}_x \cdot \mathbf{e}_\theta\right] \tag{5.308}$$

Noting that $0.42^2 = 0.18$ and that

$$\mathbf{E}_x \cdot \mathbf{e}_\theta = \mathbf{E}_x \cdot \left[-\cos\theta \, \mathbf{E}_x - \sin\theta \, \mathbf{E}_y \right] = -\cos\theta \tag{5.309}$$

we obtain

$$\frac{1}{2} m {}^{\mathcal{F}}\mathbf{v}_C \cdot {}^{\mathcal{F}}\mathbf{v}_C = \frac{1}{2} m \left[r^2 \dot{\theta}^2 + 0.18 r^2 \dot{\theta}^2 - 0.84 r^2 \dot{\theta}^2 \cos\theta \right] \tag{5.310}$$

Simplifying further, we have that

$$\frac{1}{2} m {}^{\mathcal{F}}\mathbf{v}_C \cdot {}^{\mathcal{F}}\mathbf{v}_C = \frac{1}{2} m r^2 \dot{\theta}^2 \left[1.18 - 0.84 \cos\theta \right] \tag{5.311}$$

Next, the angular momentum of the cylinder relative to the center of mass C is given as

$$^{\mathcal{F}}\mathbf{H}_C = \mathbf{I}_C^{\mathcal{R}} \cdot {}^{\mathcal{F}}\boldsymbol{\omega}^{\mathcal{R}} \tag{5.312}$$

Now since the cylinder is symmetric about the $\mathbf{e}_r$-direction, we have that $\{\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_z\}$ is a principal-axis basis. Consequently, the moment of inertia tensor $\mathbf{I}_C^{\mathcal{R}}$ can be expressed as

$$\mathbf{I}_C^{\mathcal{R}} = I_{rr}^C \mathbf{e}_r \otimes \mathbf{e}_r + I_{\theta\theta}^C \mathbf{e}_\theta \otimes \mathbf{e}_\theta + I_{zz}^C \mathbf{e}_z \otimes \mathbf{e}_z \tag{5.313}$$

Then, using the expression for ${}^{\mathcal{F}}\boldsymbol{\omega}^{\mathcal{R}}$ from Eq. (5.292), we obtain ${}^{\mathcal{F}}\mathbf{H}_C$ as

$$^{\mathcal{F}}\mathbf{H}_C = \left(I_{rr}^C \mathbf{e}_r \otimes \mathbf{e}_r + I_{\theta\theta}^C \mathbf{e}_\theta \otimes \mathbf{e}_\theta + I_{zz}^C \mathbf{e}_z \otimes \mathbf{e}_z \right) \cdot \dot{\theta} \mathbf{e}_z = I_{zz}^C \dot{\theta} \mathbf{e}_z \tag{5.314}$$

Now for a semicircular cylinder we have that

$$I_{zz}^C = 0.32 m r^2 \tag{5.315}$$

Consequently, the angular momentum of the cylinder relative to the center of mass of the cylinder is given as

$$^{\mathcal{F}}\mathbf{H}_C = 0.32 m r^2 \dot{\theta} \mathbf{E}_z \tag{5.316}$$

Using the expression for ${}^{\mathcal{F}}\mathbf{H}_C$ from Eq. (5.316), we obtain

$$\frac{1}{2} {}^{\mathcal{F}}\mathbf{H}_C \cdot {}^{\mathcal{F}}\boldsymbol{\omega}^{\mathcal{R}} = \frac{1}{2} \left[0.32 m r^2 \dot{\theta} \mathbf{E}_z \cdot \dot{\theta} \mathbf{E}_z \right] = 0.16 m r^2 \dot{\theta}^2 \tag{5.317}$$

The kinetic energy of the cylinder in reference frame $\mathcal{F}$ is then given as

$$^{\mathcal{F}}T = \frac{1}{2} m r^2 \dot{\theta}^2 \left[1.18 - 0.84 \cos\theta \right] + 0.16 m r^2 \dot{\theta}^2 \tag{5.318}$$

Eq. (5.318) simplifies to

$$^{\mathcal{F}}T = \frac{1}{2} m r^2 \dot{\theta}^2 \left[1.5 - 0.84 \cos\theta \right] \tag{5.319}$$

Now since the only conservative force acting on the cylinder is that due to gravity, the potential energy of the cylinder in reference frame $\mathcal{F}$ is given as

$$^{\mathcal{F}}U = -m\mathbf{g}\cdot\mathbf{r}_C \tag{5.320}$$

Now since the force of gravity acts vertically downward, we have that

$$m\mathbf{g} = mg\mathbf{E}_y \tag{5.321}$$

Then, using the expression for $\mathbf{r}_C$ from Eq. (5.299), we obtain $^{\mathcal{F}}U$ as

$$^{\mathcal{F}}U = -mg\mathbf{E}_y\cdot(x\mathbf{E}_x + 0.42r\mathbf{e}_r) = -0.42mgr\mathbf{E}_y\cdot\mathbf{e}_r \tag{5.322}$$

Now, using the expression for $\mathbf{e}_r$ from Eq. (5.291), we have that

$$\mathbf{E}_y\cdot\mathbf{e}_r = \mathbf{E}_y\cdot(-\sin\theta\,\mathbf{E}_x + \cos\theta\,\mathbf{E}_y) = \cos\theta \tag{5.323}$$

Consequently, the potential energy in reference frame $\mathcal{F}$ is given as

$$^{\mathcal{F}}U = -0.42mgr\cos\theta \tag{5.324}$$

The total energy of the system in reference frame $\mathcal{F}$ is then given as

$$^{\mathcal{F}}E = {^{\mathcal{F}}T} + {^{\mathcal{F}}U} = \frac{1}{2}mr^2\dot{\theta}^2\left[1.5 - 0.84\cos\theta\right] - 0.42mgr\cos\theta \tag{5.325}$$

Differentiating $^{\mathcal{F}}E$ with respect to time, we obtain

$$\frac{d}{dt}\left({^{\mathcal{F}}E}\right) = mr^2\dot{\theta}\ddot{\theta}\left[1.5 - 0.84\cos\theta\right] + \frac{1}{2}mr^2\dot{\theta}^2\left[0.84\dot{\theta}\sin\theta\right] + 0.42mgr\dot{\theta}\sin\theta = 0 \tag{5.326}$$

Simplifying this last expression gives

$$mr^2\dot{\theta}\ddot{\theta}\left[1.5 - 0.84\cos\theta\right] + 0.42mr^2\dot{\theta}^2\left[\dot{\theta}\sin\theta\right] + 0.42mgr\dot{\theta}\sin\theta = 0 \tag{5.327}$$

We then obtain

$$\dot{\theta}\left\{mr^2\ddot{\theta}\left[1.5 - 0.84\cos\theta\right] + 0.42mr^2\dot{\theta}^2\sin\theta + 0.42mgr\sin\theta\right\} = 0 \tag{5.328}$$

Noting that $\dot{\theta} \neq 0$ as a function of time, we obtain the differential equation of motion as

$$mr^2\ddot{\theta}\left[1.5 - 0.84\cos\theta\right] + 0.42mr^2\dot{\theta}^2\sin\theta + 0.42mgr\sin\theta = 0 \tag{5.329}$$

Question 5–8

A homogeneous sphere of radius r rolls without slip along a fixed spherical surface of radius R as shown in Fig. P5-8. The angle θ measures the amount by which the sphere has rotated from the vertical direction. Knowing that gravity acts downward and assuming the initial conditions $\theta(0) = 0$ and $\dot{\theta}(0) = 0$, determine the differential equation of motion while the sphere maintains contact with the spherical surface.

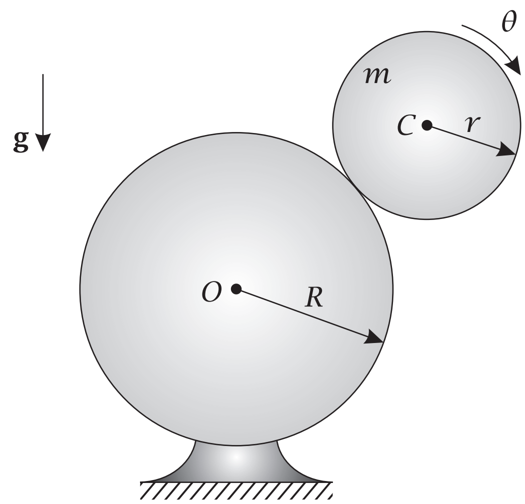

Figure P5-12

Solution to Question 5–8

Preliminaries

For this problem it is convenient to apply the following balance laws:

- Euler's 1^{st} law to the center of mass of the rolling sphere
- Euler's 2^{nd} law about the center of mass of the rolling sphere

In order to use the aforementioned balance laws, we will need the following kinematic quantities in an inertial reference frame:

- The acceleration of the center of mass of the rolling sphere
- The rate of change of angular momentum of the rolling sphere about the center of mass of the rolling sphere

Kinematics

Let $\mathcal{F}$ be a fixed reference frame. Then, choose the following coordinate system fixed in reference frame $\mathcal{F}$:

$$\begin{array}{lcl} & \text{Origin at } O & \\ \mathbf{E}_x & = & \text{Along } OC \text{ When } \theta = 0 \\ \mathbf{E}_z & = & \text{Into Page} \\ \mathbf{E}_y & = & \mathbf{E}_z \times \mathbf{E}_x \end{array}$$

Next, let $\mathcal{A}$ be a reference frame fixed to the direction OC. Then, choose the following coordinate system fixed in reference frame $\mathcal{A}$:

$$\begin{array}{lcl} & \text{Origin at } O & \\ \mathbf{e}_r & = & \text{Along } OC \\ \mathbf{e}_z & = & \text{Into Page } (= \mathbf{E}_z) \\ \mathbf{e}_\phi & = & \mathbf{e}_z \times \mathbf{e}_r \end{array}$$

Third, let $\mathcal{D}$ be the rolling sphere. Then, choose the following coordinate system fixed in reference frame $\mathcal{D}$:

$$\begin{array}{lcl} & \text{Origin at } C & \\ \mathbf{u}_r & = & \text{Fixed to } \mathcal{D} \\ \mathbf{u}_z & = & \mathbf{e}_z \\ \mathbf{u}_\theta & = & \mathbf{u}_z \times \mathbf{u}_r \end{array}$$

The geometry of the bases $\{\mathbf{E}_x, \mathbf{E}_y, \mathbf{E}_z\}$ and $\{\mathbf{e}_r, \mathbf{e}_\phi, \mathbf{e}_z\}$ is shown in Fig. 5-14 where ϕ be the angle between the direction $\mathbf{E}_x$ and the direction $\mathbf{e}_r$. Using the geometry in

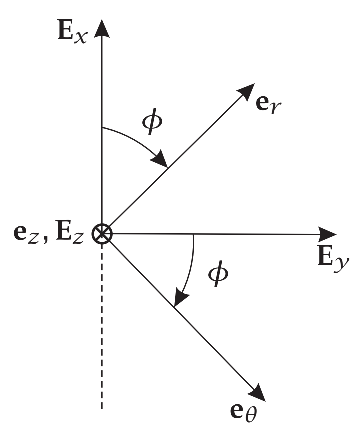

Figure 5-14 Relationship Between Bases $\{\mathbf{E}_x, \mathbf{E}_y, \mathbf{E}_z\}$ and $\{\mathbf{e}_r, \mathbf{e}_\phi, \mathbf{e}_z\}$ for Question 5–8.

Fig. 5-15, we have the following relationship between $\{\mathbf{E}_x, \mathbf{E}_y, \mathbf{E}_z\}$ and $\{\mathbf{e}_r, \mathbf{e}_\phi, \mathbf{e}_z\}$:

$$\begin{aligned} \mathbf{e}_r &= \cos\phi \mathbf{E}_x + \sin\phi \mathbf{E}_y \\ \mathbf{e}_\phi &= -\sin\phi \mathbf{E}_x + \cos\phi \mathbf{E}_y \\ \mathbf{E}_x &= \cos\phi \mathbf{e}_r - \sin\phi \mathbf{e}_\phi \\ \mathbf{E}_y &= \sin\phi \mathbf{e}_r + \cos\phi \mathbf{e}_\phi \end{aligned} \tag{5.330}$$

Next, the angular velocity of reference frame $\mathcal{A}$ in reference frame $\mathcal{F}$ is given as

$$^{\mathcal{F}}\boldsymbol{\omega}^{\mathcal{A}} = \dot{\phi}\mathbf{e}_z = \dot{\phi}\mathbf{E}_z \tag{5.331}$$

Furthermore, denoting the reference frame of the rolling sphere by $\mathcal{R}$ and observing that θ describes the rotation of the rolling sphere relative to the fixed vertical direction, we have that

$$^{\mathcal{F}}\boldsymbol{\omega}^{\mathcal{R}} = \dot{\theta}\mathbf{e}_z = \dot{\theta}\mathbf{E}_z \tag{5.332}$$

The position of the center of mass of the rolling sphere is then given as

$$\bar{\mathbf{r}} = (R + r)\mathbf{e}_r \tag{5.333}$$

Differentiating $\bar{\mathbf{r}}$ in reference frame $\mathcal{F}$ using the transport theorem, we have that

$$^{\mathcal{F}}\bar{\mathbf{v}} = \frac{^{\mathcal{F}}d}{dt}(\bar{\mathbf{r}}) = \frac{^{\mathcal{A}}d}{dt}(\bar{\mathbf{r}}) + {^{\mathcal{F}}\boldsymbol{\omega}^{\mathcal{A}}} \times \bar{\mathbf{r}} \tag{5.334}$$

Now we have that

$$\begin{aligned} \frac{^{\mathcal{A}}d}{dt}(\bar{\mathbf{r}}) &= \mathbf{0} \\ ^{\mathcal{F}}\boldsymbol{\omega}^{\mathcal{A}} \times \bar{\mathbf{r}} &= \dot{\theta}\mathbf{e}_z \times (R + r)\mathbf{e}_r = (R + r)\dot{\phi}\mathbf{e}_\phi \end{aligned} \tag{5.335}$$

Consequently,

$$^{\mathcal{F}}\bar{\mathbf{v}} = (R + r)\dot{\phi}\mathbf{e}_\phi \tag{5.336}$$

Differentiating $^{\mathcal{F}}\bar{\mathbf{v}}$ in reference frame $\mathcal{F}$ using the transport theorem, we have that

$$^{\mathcal{F}}\bar{\mathbf{a}} = \frac{^{\mathcal{F}}d}{dt}\left({^{\mathcal{F}}\bar{\mathbf{v}}}\right) = \frac{^{\mathcal{A}}d}{dt}\left({^{\mathcal{F}}\bar{\mathbf{v}}}\right) + {^{\mathcal{F}}\boldsymbol{\omega}^{\mathcal{A}}} \times {^{\mathcal{F}}\bar{\mathbf{v}}} \tag{5.337}$$

Now we have that

$$\begin{aligned} \frac{^{\mathcal{A}}d}{dt}\left({^{\mathcal{F}}\bar{\mathbf{v}}}\right) &= (R + r)\ddot{\phi}\mathbf{e}_\phi \\ ^{\mathcal{F}}\boldsymbol{\omega}^{\mathcal{A}} \times \bar{\mathbf{r}} &= \dot{\theta}\mathbf{e}_z \times (R + r)\dot{\phi}\mathbf{e}_\phi = -(R + r)\dot{\phi}^2\mathbf{e}_r \end{aligned} \tag{5.338}$$

Consequently,

$$^{\mathcal{F}}\bar{\mathbf{a}} = -(R + r)\dot{\phi}^2\mathbf{e}_r + (R + r)\ddot{\phi}\mathbf{e}_\phi \tag{5.339}$$

Now we see that Eq. (5.339) is an expression for ${}^{\mathcal{F}}\bar{\mathbf{a}}$ in terms of the derivatives of ϕ. However, in this problem we are interested in obtaining the differential equation of motion in terms of θ. Therefore, we need to eliminate ϕ in favor of θ.

Eliminating ϕ is accomplished as follows. First, we know that the sphere rolls without slip along the fixed sphere. Denoting Q as the instantaneous point of contact between the two spheres, we have that

$$ {}^{\mathcal{F}}\mathbf{v}_Q^{\mathcal{R}} \equiv {}^{\mathcal{F}}\mathbf{v}_Q = \mathbf{0} \tag{5.340} $$

Then, applying the relative velocity property for two points on a rigid body, we have that

$$ {}^{\mathcal{F}}\bar{\mathbf{v}} - {}^{\mathcal{F}}\mathbf{v}_Q = {}^{\mathcal{F}}\boldsymbol{\omega}^{\mathcal{R}} \times (\bar{\mathbf{r}} - \mathbf{f}_Q) \tag{5.341} $$

Now we have that $\mathbf{r}_Q = R\mathbf{e}_r$ which implies that $\bar{\mathbf{r}} - \mathbf{r}_Q = r\mathbf{e}_r$. Then, substituting ${}^{\mathcal{F}}\boldsymbol{\omega}^{\mathcal{R}}$ from Eq. (5.332) into Eq. (5.341), we obtain

$$ {}^{\mathcal{F}}\bar{\mathbf{v}} - {}^{\mathcal{F}}\mathbf{v}_Q = \dot{\theta}\mathbf{e}_z \times r\mathbf{e}_r = r\dot{\theta}\mathbf{e}_\phi \tag{5.342} $$

Finally, observing from Eq. (5.340) that ${}^{\mathcal{F}}\mathbf{v}_Q = \mathbf{0}$, we obtain ${}^{\mathcal{F}}\bar{\mathbf{v}}$ as

$$ {}^{\mathcal{F}}\bar{\mathbf{v}} = r\dot{\theta}\mathbf{e}_\phi \tag{5.343} $$

Then, setting the result of Eq. (5.343) equal to the result of Eq. (5.336), we obtain

$$ r\dot{\theta} = (R + r)\dot{\phi} \tag{5.344} $$

Solving Eq. (5.344) for $\dot{\phi}$, we obtain

$$ \dot{\phi} = \frac{r}{R + r}\dot{\theta} \tag{5.345} $$

Differentiating $\dot{\phi}$ in Eq. (5.345), we obtain

$$ \ddot{\phi} = \frac{r}{R + r}\ddot{\theta} \tag{5.346} $$

The acceleration of the center of mass of the rolling sphere is then given as

$$ {}^{\mathcal{F}}\bar{\mathbf{a}} = -(R + r)\left[\frac{r}{R + r}\dot{\theta}\right]^2 \mathbf{e}_r + (R + r)\left[\frac{r}{R + r}\ddot{\theta}\right]\mathbf{e}_\phi \tag{5.347} $$

Simplifying Eq. (5.347), we obtain

$$ {}^{\mathcal{F}}\bar{\mathbf{a}} = -\frac{r^2\dot{\theta}^2}{R + r}\mathbf{e}_r + r\ddot{\theta}\mathbf{e}_\phi \tag{5.348} $$

Next, we need to compute the rate of change of the angular momentum of the sphere relative to the center of mass of the sphere in reference frame $\mathcal{F}$. We have that

$$ {}^{\mathcal{F}}\bar{\mathbf{H}} = \bar{\mathbf{I}}^{\mathcal{R}} \cdot {}^{\mathcal{F}}\boldsymbol{\omega}^{\mathcal{R}} \tag{5.349} $$

where $\bar{\mathbf{I}}^{\mathcal{R}}$ is the moment of inertia tensor relative to the center of mass and ${}^{\mathcal{F}}\boldsymbol{\omega}^{\mathcal{R}}$ is the angular velocity of the sphere in reference frame $\mathcal{F}$. Now since $\{\mathbf{e}_r, \mathbf{e}_\phi, \mathbf{e}_z\}$ is a principal-axis basis, the moment of inertia tensor $\bar{\mathbf{I}}^{\mathcal{R}}$ can be written as

$$\bar{\mathbf{I}}^{\mathcal{R}} = \bar{I}_{rr}\mathbf{u}_r \otimes \mathbf{u}_r + \bar{I}_{\theta\theta}\mathbf{u}_\theta \otimes \mathbf{u}_\theta + \bar{I}_{zz}\mathbf{u}_z \otimes \mathbf{u}_z \tag{5.350}$$

Substituting the expression for ${}^{\mathcal{F}}\boldsymbol{\omega}^{\mathcal{R}}$ as given in Eq. (5.332) and the expression for the moment of inertia tensor from Eq. (5.350) into Eq. (5.349), we obtain ${}^{\mathcal{F}}\bar{\mathbf{H}}$ as

$${}^{\mathcal{F}}\bar{\mathbf{H}} = \bar{I}_{zz}\dot{\theta}\mathbf{u}_z \tag{5.351}$$

Now we have for a sphere that

$$\bar{I}_{zz} = \frac{2}{5}mr^2 \tag{5.352}$$

Consequently, Eq. (5.351) simplifies to

$${}^{\mathcal{F}}\bar{\mathbf{H}} = \frac{2}{5}mr^2\dot{\theta}\mathbf{u}_z \tag{5.353}$$

Differentiating ${}^{\mathcal{F}}\bar{\mathbf{H}}$ in reference frame $\mathcal{F}$, we obtain

$$\frac{{}^{\mathcal{F}}d}{dt}\left({}^{\mathcal{F}}\bar{\mathbf{H}}\right) = \frac{2}{5}mr^2\ddot{\theta}\mathbf{u}_z \tag{5.354}$$

Kinetics

As stated earlier, this problem will be solved using the following balance laws:

- Euler's 1^{st} law to the rolling sphere
- Euler's 2^{nd} law about the center of mass of the rolling sphere

The free body diagram of the rolling sphere is shown in Fig. 5-15. It can be seen that the following forces act on the sphere:

$$\begin{array}{lcl} \mathbf{F}_r & = & \text{Force of Rolling} \\ \mathbf{N} & = & \text{Reaction Force of Fixed Sphere on Rolling Sphere} \\ m\mathbf{g} & = & \text{Force of Gravity} \end{array}$$

Given the geometry of the problem, we have that

$$\begin{array}{lcl} \mathbf{N} & = & N\mathbf{e}_r \\ \mathbf{F}_r & = & F_r\mathbf{e}_\phi \\ m\mathbf{g} & = & -mg\mathbf{E}_x \end{array} \tag{5.355}$$

Then, substituting the expression for $\mathbf{E}_x$ in terms of $\mathbf{e}_r$ and $\mathbf{e}_\phi$ from Eq. (5.330), the force of gravity can be written as

$$m\mathbf{g} = -mg\cos\phi\mathbf{e}_r + mg\sin\phi\mathbf{e}_\phi \tag{5.356}$$

Then, using the fact that θ is zero when the sphere is at the top of the fixed sphere, we have that

$$\theta(t=0) = \phi(t=0) = 0 \tag{5.357}$$

Consequently, integrating Eq. (5.345), we obtain

$$\phi = \frac{r\theta}{R+r} \tag{5.358}$$

Substituting the expression for ϕ into Eq. (5.356), we obtain

$$m\mathbf{g} = -mg\cos\left(\frac{r\theta}{R+r}\right) + mg\sin\left(\frac{r\theta}{R+r}\right)\mathbf{e}_\phi \tag{5.359}$$

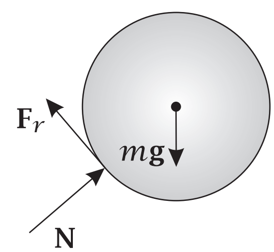

Figure 5-15 Free Body Diagram of Rolling Sphere for Question 5–8.

Application of Euler's 1^{st} ***To the Rolling Sphere***

Euler's 1^{st} law states that

$$\mathbf{F} = m^{\mathcal{F}}\bar{\mathbf{a}} \tag{5.360}$$

For this problem, the resultant force acting on the sphere is given as

$$\mathbf{F} = \mathbf{N} + \mathbf{F}_r + m\mathbf{g} \tag{5.361}$$

Using the expressions for the forces as given in Eq. (5.355) and Eq. (5.359), we have the resultant force as

$$\mathbf{F} = N\mathbf{e}_r + F_r\mathbf{e}_\phi - mg\cos\left(\frac{r}{R+r}\theta\right) + mg\sin\left(\frac{r}{R+r}\theta\right)\mathbf{e}_\phi \tag{5.362}$$

Then, setting $\mathbf{F}$ from Eq. (5.362) equal to $m^{\mathcal{F}}\bar{\mathbf{a}}$ using ${}^{\mathcal{F}}\bar{\mathbf{a}}$ from Eq. (5.348), we obtain

$$N\mathbf{e}_r + F_r\mathbf{e}_\phi - mg\cos\left(\frac{r\theta}{R+r}\right) + mg\sin\left(\frac{r\theta}{R+r}\right)\mathbf{e}_\phi = m\left(-\frac{r^2\dot{\theta}^2}{R+r}\mathbf{e}_r + r\ddot{\theta}\mathbf{e}_\phi\right) \tag{5.363}$$

Simplifying Eq. (5.363), we obtain

$$\left[N - mg\cos\left(\frac{r\theta}{R+r}\right)\right]\mathbf{e}_r + \left[F_r + mg\sin\left(\frac{r\theta}{R+r}\right)\right] = -m\frac{r^2\dot{\theta}^2}{R+r}\mathbf{e}_r + mr\ddot{\theta}\mathbf{e}_\phi \tag{5.364}$$

Eq. (5.364) yields the following two scalar equations:

$$N - mg\cos\left(\frac{r\theta}{R+r}\right) = -m\frac{r^2\dot{\theta}^2}{R+r} \tag{5.365}$$

$$F_r + mg\sin\left(\frac{r\theta}{R+r}\right) = mr\ddot{\theta} \tag{5.366}$$

Application of Euler's 2^{nd} About the Center of Mass of the Rolling Sphere

We recall Euler's 2^{nd} law relative to the center of mass of a rigid body as

$$\bar{\mathbf{M}} = \frac{{}^{\mathcal{F}}d}{dt}\left({}^{\mathcal{F}}\bar{\mathbf{H}}\right) \tag{5.367}$$

We already have ${}^{\mathcal{F}}d\left({}^{\mathcal{F}}\bar{\mathbf{H}}\right)/dt$ from Eq. (5.354). Next, observing that $m\mathbf{g}$ and $\mathbf{N}$ both pass through the center of mass of the sphere, we obtain $\bar{\mathbf{M}}$ as

$$\bar{\mathbf{M}} = (\mathbf{r}_Q - \bar{\mathbf{r}}) \times \mathbf{F}_r \tag{5.368}$$

Then, using the fact that $\mathbf{r}_Q - \bar{\mathbf{r}} = -r\mathbf{e}_r$ and the expression for $\mathbf{F}_r$ from Eq. (5.355), we obtain $\bar{\mathbf{M}}$ as

$$\bar{\mathbf{M}} = -r\mathbf{e}_r \times F_r\mathbf{e}_\phi = -rF_r\mathbf{e}_z = -rF_r\mathbf{u}_z \tag{5.369}$$

where we note that $\mathbf{e}_z = \mathbf{u}_z$. Then, setting $\bar{\mathbf{M}}$ from Eq. (5.369) equal to ${}^{\mathcal{F}}d\left({}^{\mathcal{F}}\bar{\mathbf{H}}\right)/dt$ using the expression for ${}^{\mathcal{F}}d\left({}^{\mathcal{F}}\bar{\mathbf{H}}\right)/dt$ from Eq. (5.354), we obtain

$$-rF_r = \frac{2}{5}mr^2\ddot{\theta} \tag{5.370}$$

Solving Eq. (5.370) for F_r, we obtain

$$F_r = -\frac{2}{5}mr\ddot{\theta} \tag{5.371}$$

Differential Equation of Motion

The differential equation of motion can now be found using Eq. (5.365) and Eq. (5.371). In particular, substituting F_r from Eq. (5.371) into Eq. (5.365), we obtain

$$-\frac{2}{5}mr\ddot{\theta} + mg\sin\left(\frac{r\theta}{R+r}\right) = mr\ddot{\theta} \tag{5.372}$$

Rearranging Eq. (5.372), we obtain

$$\frac{7}{5}mr\ddot{\theta} - mg\sin\left(\frac{r\theta}{R+r}\right) = 0 \tag{5.373}$$

Simplifying Eq. (5.373), we obtain

$$\ddot{\theta} - \frac{5g}{7r}\sin\left(\frac{r\theta}{R+r}\right) = 0 \tag{5.374}$$

Question 5–10

A uniform circular disk of mass m and radius r rolls without slip along a plane inclined at a constant angle β with horizontal as shown in Fig. P5-10. Attached at the center of the disk is a linear spring with spring constant K. Knowing that the spring is unstretched when the angle θ is zero and that gravity acts downward, determine the differential equation of motion for the disk in terms of the angle θ.

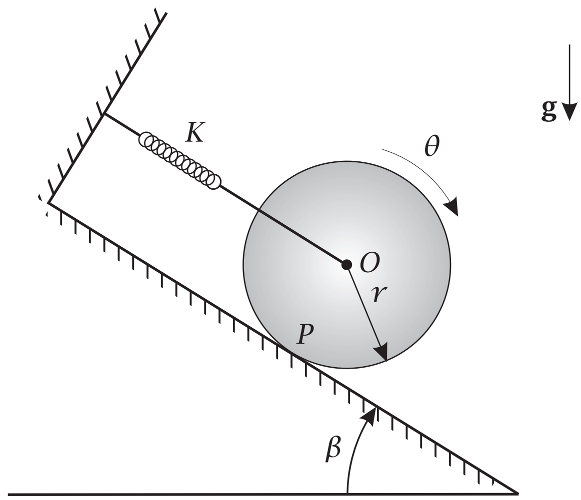

Figure P5-13

Solution to Question 5–10

Kinematics

First, let $\mathcal{F}$ be a fixed reference frame. Then, choose the following coordinate system fixed in $\mathcal{F}$:

$$
\begin{array}{ccr}
 & \text{Origin at Point } O \text{ when } t = 0 & \\
\mathbf{E}_x & = & \text{Down Incline} \\
\mathbf{E}_z & = & \text{Into Page} \\
\mathbf{E}_y & = & \mathbf{E}_z \times \mathbf{E}_x
\end{array}
$$

Next, let $\mathcal{D}$ be the disk. Then, choose the following coordinate system fixed in $\mathcal{D}$:

$$
\begin{array}{ccr}
 & \text{Origin at } O & \\
\mathbf{e}_r & = & \text{Fixed to } \mathcal{D} \\
\mathbf{e}_z & = & \mathbf{E}_z \\
\mathbf{e}_\theta & = & \mathbf{e}_z \times \mathbf{e}_r
\end{array}
$$

Now because the disk rolls without slip along the incline, we have

$$
{}^{\mathcal{F}}\mathbf{v}_P^{\mathcal{D}} = {}^{\mathcal{F}}\mathbf{v}_P^{\mathcal{F}} = \mathbf{0} \tag{5.375}
$$

The velocity of the center of mass of the disk is then obtained as

$$^{\mathcal{F}}\bar{\mathbf{v}} = {}^{\mathcal{F}}\mathbf{v}_P + {}^{\mathcal{F}}\boldsymbol{\omega}^{\mathcal{R}} \times (\bar{\mathbf{r}} - \mathbf{r}_P) \tag{5.376}$$

In terms of the basis $\{\mathbf{E}_x, \mathbf{E}_y, \mathbf{E}_z\}$, the angular velocity of the disk in reference frame $\mathcal{F}$ is given as

$$^{\mathcal{F}}\boldsymbol{\omega}^{\mathcal{R}} = \dot{\theta}\mathbf{E}_z \tag{5.377}$$

$$\bar{\mathbf{r}} - \mathbf{r}_P = -r\mathbf{E}_y \tag{5.378}$$

where $\mathcal{R}$ denotes the reference frame of the disk. Consequently, the velocity of the center of mass of the disk in reference frame $\mathcal{F}$ is given as

$$^{\mathcal{F}}\bar{\mathbf{v}} = \dot{\theta}\mathbf{E}_z \times (-r\mathbf{E}_y) = r\dot{\theta}\mathbf{E}_x \tag{5.379}$$

Computing the rate of change of $^{\mathcal{F}}\bar{\mathbf{v}}$ in reference frame $\mathcal{F}$, we obtain the acceleration of the center of mass of the disk as

$$^{\mathcal{F}}\bar{\mathbf{a}} = \frac{^{\mathcal{F}}d}{dt}\left({}^{\mathcal{F}}\bar{\mathbf{v}}\right) r\ddot{\theta}\mathbf{E}_x \tag{5.380}$$

Next, the angular momentum relative to the instantaneous point of contact is given as

$$^{\mathcal{F}}\mathbf{H}_P = {}^{\mathcal{F}}\bar{\mathbf{H}} + (\mathbf{r}_Q - \bar{\mathbf{r}}) \times m({}^{\mathcal{F}}\mathbf{v}_Q - {}^{\mathcal{F}}\bar{\mathbf{v}}) \tag{5.381}$$

Now we have

$$^{\mathcal{F}}\bar{\mathbf{H}} = \bar{\mathbf{I}}^{\mathcal{R}} \cdot {}^{\mathcal{F}}\boldsymbol{\omega}^{\mathcal{R}} \tag{5.382}$$

Now because $\{\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_z\}$ is a principal-axis basis, we have

$$\bar{\mathbf{I}}^{\mathcal{R}} = \bar{I}_{rr}\mathbf{e}_r \otimes \mathbf{e}_r + \bar{I}_{\theta\theta}\mathbf{e}_\theta \otimes \mathbf{e}_\theta + \bar{I}_{zz}\mathbf{e}_z \otimes \mathbf{e}_z \tag{5.383}$$

Substituting $\bar{\mathbf{I}}^{\mathcal{R}}$ from Eq. (5.383) into Eq. (5.382), we obtain

$$^{\mathcal{F}}\bar{\mathbf{H}} = \bar{I}_{zz}\dot{\theta}\mathbf{e}_z \tag{5.384}$$

Now for a uniform circular disk we have $\bar{I}_{zz} = mr^2/2$. Therefore,

$$^{\mathcal{F}}\bar{\mathbf{H}} = \frac{mr^2}{2}\dot{\theta}\mathbf{e}_z \tag{5.385}$$

Next, since $^{\mathcal{F}}\mathbf{v}_Q = \mathbf{0}$, we have

$$(\mathbf{r}_Q - \mathbf{r}_O) \times m({}^{\mathcal{F}}\mathbf{v}_Q - {}^{\mathcal{F}}\mathbf{v}_O) = mr\mathbf{E}_y \times (-r\dot{\theta}\mathbf{E}_x) = mr^2\dot{\theta}\mathbf{E}_z = mr^2\dot{\theta}\mathbf{e}_z \tag{5.386}$$

Consequently,

$$^{\mathcal{F}}\mathbf{H}_Q = \frac{mr^2}{2}\dot{\theta}\mathbf{e}_z + mr^2\dot{\theta}\mathbf{e}_z = \frac{3}{2}mr^2\dot{\theta}\mathbf{e}_z \tag{5.387}$$

The rate of change of angular momentum relative to the point of contact is then given as

$$\frac{^{\mathcal{F}}d}{dt}\left({}^{\mathcal{F}}\mathbf{H}_Q\right) = \frac{3}{2}mr^2\ddot{\theta}\mathbf{e}_z \tag{5.388}$$

Kinetics

The free body diagram of the cylinder is shown in Fig. 5-16.

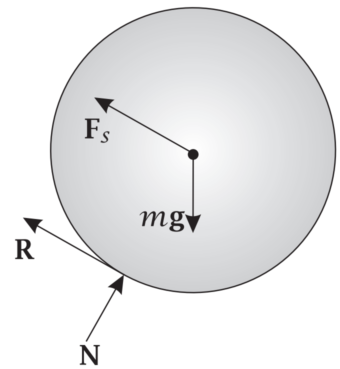

Figure 5-16 Free Body Diagram of Cylinder for Question 5–10.

Using Fig. 5-16, the forces acting on the cylinder are given as

$$\begin{array}{lcl} \mathbf{N} & = & \text{Force of Incline on Cylinder} \\ \mathbf{R} & = & \text{Force of Rolling} \\ \mathbf{F}_s & = & \text{Spring Force} \\ m\mathbf{g} & = & \text{Force of Gravity} \end{array}$$

From the geometry we have that

$$\mathbf{N} = N\mathbf{E}_y \tag{5.389}$$
$$\mathbf{R} = R\mathbf{E}_x \tag{5.390}$$
$$\mathbf{F}_s = -K(\ell - \ell_0)\mathbf{u}_s \tag{5.391}$$
$$m\mathbf{g} = mg\mathbf{u}_v \tag{5.392}$$
$$\tag{5.393}$$

where $\mathbf{u}_v$ is the unit vector in the vertically downward direction and $\mathbf{u}_s$ is the direction of the spring force. Now $\mathbf{u}_v$ is shown in Fig. 5-17.

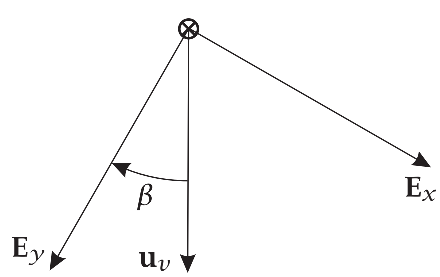

Figure 5-17 Unit Vector in Vertically Downward Direction for Question 5–10.

Using Fig. 5-17, we have that

$$\mathbf{u}_v = \sin\beta\mathbf{E}_x + \cos\beta\mathbf{E}_y \tag{5.394}$$

Therefore, the force of gravity is obtained as

$$m\mathbf{g} = mg\sin\beta\mathbf{E}_x + mg\cos\beta\mathbf{E}_y \tag{5.395}$$

Next, we have for the spring force that

$$\ell = \|\bar{\mathbf{r}} - \mathbf{r}_A\| \tag{5.396}$$

where A is the attachment point of the spring. Now suppose we say that the spring is attached a distance L from where O is located when $\theta = 0$. Then, we have that

$$\mathbf{r}_A = -L\mathbf{E}_x \tag{5.397}$$

Then we have that

$$\bar{\mathbf{r}} - \mathbf{r}_A = x\mathbf{E}_x - (-L\mathbf{E}_x) = (x + L)\mathbf{E}_x \tag{5.398}$$

Furthermore, the unstretched length of the spring is given as

$$\ell_0 = L \tag{5.399}$$

Consequently, we obtain

$$\|\bar{\mathbf{r}} - \mathbf{r}_A\| - L = x \tag{5.400}$$

Now since the disk rolls without slip, we have that

$$x = r\theta \tag{5.401}$$

which implies that

$$\|\bar{\mathbf{r}} - \mathbf{r}_A\| - L = r\theta \tag{5.402}$$

Finally, the direction $\mathbf{u}_s$ is given as

$$\mathbf{u}_s = \frac{\bar{\mathbf{r}} - \mathbf{r}_A}{\|\bar{\mathbf{r}} - \mathbf{r}_A\|} = \mathbf{E}_x \tag{5.403}$$

The spring force is then given as

$$\mathbf{F}_s = -Kr\theta\mathbf{E}_x \tag{5.404}$$

Using the forces above, we can now apply the general form of Euler's 2^{nd} law to the disk, i.e.,

$$\mathbf{M}_Q - (\bar{\mathbf{r}} - \mathbf{r}_Q) \times m^{\mathcal{F}}\mathbf{a}_Q = \frac{^{\mathcal{F}}d}{dt}\left(^{\mathcal{F}}\mathbf{H}_Q\right) \tag{5.405}$$

First, since $\mathbf{N}$ and $\mathbf{R}$ pass through point Q, the moment relative to point Q is given as

$$\mathbf{M}_Q = (\bar{\mathbf{r}} - \mathbf{r}_Q) \times (m\mathbf{g} + \mathbf{F}_s) \tag{5.406}$$

where we note that both the force of gravity and the spring force act at the center of mass. Then, using the expressions for $m\mathbf{g}$ and $\mathbf{F}_s$ from above, we obtain

$$\begin{aligned}\mathbf{M}_Q &= (-r\mathbf{E}_y) \times (mg\sin\beta\mathbf{E}_x + mg\cos\beta\mathbf{E}_y - Kr\theta\mathbf{E}_x) \\ &= mgr\sin\beta\mathbf{E}_z - Kr^2\theta\mathbf{E}_z \\ &= (mgr\sin\beta - Kr^2\theta)\mathbf{E}_z\end{aligned} \tag{5.407}$$

Next, we have

$$^{\mathcal{F}}\mathbf{a}_Q = {}^{\mathcal{F}}\bar{\mathbf{a}} + {}^{\mathcal{F}}\boldsymbol{\alpha}^{\mathcal{R}} \times (\mathbf{r}_Q - \mathbf{r}_O) + {}^{\mathcal{F}}\boldsymbol{\omega}^{\mathcal{R}} \times [\boldsymbol{\omega} \times (\mathbf{r}_Q - \mathbf{r}_O)] \tag{5.408}$$

Using ${}^{\mathcal{F}}\bar{\mathbf{a}}$ from Eq. (5.380) and the fact that $\mathbf{r}_Q - \bar{\mathbf{r}} = r\mathbf{E}_y$, we obtain ${}^{\mathcal{F}}\mathbf{a}_Q$ as

$$\begin{aligned}{}^{\mathcal{F}}\mathbf{a}_Q &= r\ddot{\theta}\mathbf{E}_x + \ddot{\theta}\mathbf{E}_z \times r\mathbf{E}_y + \dot{\theta}\mathbf{E}_z \times \left[\dot{\theta}\mathbf{E}_z \times r\mathbf{E}_y\right] \\ &= r\ddot{\theta}\mathbf{E}_x - r\ddot{\theta}\mathbf{E}_x - r\dot{\theta}^2\mathbf{E}_y \\ &= -r\dot{\theta}^2\mathbf{E}_y\end{aligned} \tag{5.409}$$

Consequently, the inertial moment $-(\bar{\mathbf{r}} - \mathbf{r}_Q) \times m{}^{\mathcal{F}}\mathbf{a}_Q$ is

$$-r\mathbf{E}_y \times m(-r\dot{\theta}^2\mathbf{E}_y) = \mathbf{0} \tag{5.410}$$

Then, since the inertial moment is zero, we can set the moment relative to Q equal to the rate of change of angular momentum relative to Q to obtain

$$mgr\sin\beta - Kr^2\theta = \frac{3}{2}mr^2\ddot{\theta} \tag{5.411}$$

Rearranging, we obtain the differential equation of motion as

$$\frac{3}{2}mr^2\ddot{\theta} + Kr^2\theta - mgr\sin\beta = 0 \tag{5.412}$$

Solution to Question 5–11

A slender rod of mass m and length l is suspended from a massless collar at point O as shown in Fig. P5-11. The collar in turn slides without friction along a horizontal track. The position of the collar is denoted as x while the angle formed by the rod with the vertical is denoted θ. Given that a *known* horizontal force $\boldsymbol{P}$ is applied to the rod at the point O and that gravity acts downward, determine a system of two differential equations describing the motion of the rod in terms of the variables x and θ.

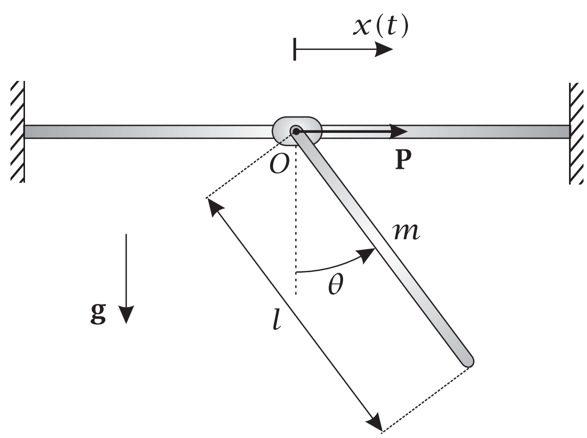

Figure P5-16

Solution to Question 5–11

Kinematics

First, let $\mathcal{F}$ be a reference frame fixed to the track. Then, choose the following coordinate system fixed in reference frame $\mathcal{F}$:

$$
\begin{array}{lcr}
 & \text{Origin at } O \text{ When } x = 0 & \\
\mathbf{E}_x & = & \text{To The Right} \\
\mathbf{E}_z & = & \text{Out of Page} \\
\mathbf{E}_y & = & \mathbf{E}_z \times \mathbf{E}_x
\end{array}
$$

Next, let $\mathcal{R}$ be a reference frame fixed to the rod. Then, choose the following coordinate system fixed in reference frame $\mathcal{R}$:

$$
\begin{array}{lcr}
 & \text{Origin at } O & \\
\mathbf{e}_r & = & \text{Along Rod} \\
\mathbf{e}_z & = & \text{Out of Page} \\
\mathbf{e}_\theta & = & \mathbf{e}_z \times \mathbf{e}_r
\end{array}
$$

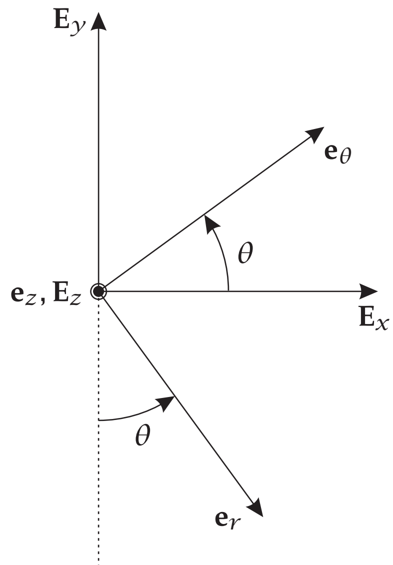

Figure 5-18 Geometry of Bases $\{\mathbf{E}_x, \mathbf{E}_y, \mathbf{E}_z\}$ and $\{\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_z\}$ for Question 5–11.

The geometry of the bases $\{\mathbf{E}_x, \mathbf{E}_y, \mathbf{E}_z\}$ and $\{\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_z\}$ is given in Fig. 5-18. Using Fig. 5-18, we have that

$$\mathbf{e}_r = \sin\theta\,\mathbf{E}_x - \cos\theta\,\mathbf{E}_y \tag{5.413}$$
$$\mathbf{e}_\theta = \cos\theta\,\mathbf{E}_x + \sin\theta\,\mathbf{E}_y \tag{5.414}$$

Furthermore, we have that

$$\mathbf{E}_x = \sin\theta\,\mathbf{e}_r + \cos\theta\,\mathbf{e}_\theta \tag{5.415}$$
$$\mathbf{E}_y = -\cos\theta\,\mathbf{e}_r + \sin\theta\,\mathbf{e}_\theta \tag{5.416}$$

The position of the center of mass of the rod is then given as

$$\bar{\mathbf{r}} = \mathbf{r}_O + \mathbf{r}_{C/O} \tag{5.417}$$

where C is the center of mass of the rod. Now we have that

$$\mathbf{r}_O = x\mathbf{E}_x \tag{5.418}$$
$$\mathbf{r}_{C/O} = \frac{l}{2}\mathbf{e}_r \tag{5.419}$$

Therefore,

$$\bar{\mathbf{r}} = x\mathbf{E}_x + \frac{l}{2}\mathbf{e}_r \tag{5.420}$$

Furthermore, the angular velocity of the rod in reference frame $\mathcal{F}$ is given as

$$^{\mathcal{F}}\boldsymbol{\omega}^{\mathcal{R}} = \dot{\theta}\mathbf{e}_z \tag{5.421}$$

Then, the velocity of the center of mass of the rod in reference frame $\mathcal{F}$ is given as

$$^{\mathcal{F}}\bar{\mathbf{v}} = \frac{^{\mathcal{F}}d\bar{\mathbf{r}}}{dt} = \frac{^{\mathcal{F}}d\mathbf{r}_O}{dt} + \frac{^{\mathcal{F}}d\mathbf{r}_{C/O}}{dt} = {^{\mathcal{F}}\mathbf{v}_O} + {^{\mathcal{F}}\mathbf{v}_{C/O}} \tag{5.422}$$

Now we have that

$$^{\mathcal{F}}\mathbf{v}_O = \frac{^{\mathcal{F}}d\mathbf{r}_O}{dt} = \dot{x}\mathbf{E}_x \tag{5.423}$$

Furthermore, using the rate of change transport theorem, we have that

$$\frac{^{\mathcal{F}}d\mathbf{r}_{C/O}}{dt} = \frac{^{\mathcal{R}}d\mathbf{r}_{C/O}}{dt} + {^{\mathcal{F}}\boldsymbol{\omega}^{\mathcal{R}}} \times \mathbf{r}_{C/O} \tag{5.424}$$

Now we have that

$$\frac{^{\mathcal{R}}d\mathbf{r}_{C/O}}{dt} = \mathbf{0} \tag{5.425}$$

$$^{\mathcal{F}}\boldsymbol{\omega}^{\mathcal{R}} \times \mathbf{r}_{C/O} = \dot{\theta}\mathbf{e}_z \times \frac{l}{2}\mathbf{e}_r = \frac{l\dot{\theta}}{2}\mathbf{e}_\theta \tag{5.426}$$

Therefore, we obtain $^{\mathcal{F}}\mathbf{v}_{C/O}$ as

$$^{\mathcal{F}}\mathbf{v}_{C/O} = \frac{l\dot{\theta}}{2}\mathbf{e}_\theta \tag{5.427}$$

which implies that

$$^{\mathcal{F}}\bar{\mathbf{v}} = \dot{x}\mathbf{E}_x + \frac{l\dot{\theta}}{2}\mathbf{e}_\theta \tag{5.428}$$

The acceleration of the center of mass is then obtained as

$$^{\mathcal{F}}\bar{\mathbf{a}} = \frac{^{\mathcal{F}}d}{dt}\left({^{\mathcal{F}}\bar{\mathbf{v}}}\right) = \frac{^{\mathcal{F}}d}{dt}\left({^{\mathcal{F}}\mathbf{v}_O}\right) + \frac{^{\mathcal{F}}d}{dt}\left({^{\mathcal{F}}\mathbf{v}_{C/O}}\right) = {^{\mathcal{F}}\mathbf{a}_O} + {^{\mathcal{F}}\mathbf{a}_{C/O}} \tag{5.429}$$

First, we obtain $^{\mathcal{F}}\mathbf{a}_O$ as

$$^{\mathcal{F}}\mathbf{a}_O = \ddot{x}\mathbf{E}_x \tag{5.430}$$

Furthermore, using the rate of change transport theorem, we have $^{\mathcal{F}}\mathbf{a}_{C/O}$ as

$$^{\mathcal{F}}\mathbf{a}_{C/O} = \frac{^{\mathcal{F}}d}{dt}\left({^{\mathcal{F}}\mathbf{v}_{C/O}}\right) = \frac{^{\mathcal{R}}d}{dt}\left({^{\mathcal{F}}\mathbf{v}_{C/O}}\right) + {^{\mathcal{F}}\boldsymbol{\omega}^{\mathcal{R}}} \times {^{\mathcal{F}}\mathbf{v}_{C/O}} \tag{5.431}$$

Now we know that

$$\frac{^{\mathcal{R}}d}{dt}\left({^{\mathcal{F}}\mathbf{v}_{C/O}}\right) = \frac{l\ddot{\theta}}{2}\mathbf{e}_\theta \tag{5.432}$$

$$^{\mathcal{F}}\boldsymbol{\omega}^{\mathcal{R}} \times {^{\mathcal{F}}\mathbf{v}_{C/O}} = \dot{\theta}\mathbf{e}_z \times \frac{l\dot{\theta}}{2}\mathbf{e}_\theta = -\frac{l\dot{\theta}^2}{2}\mathbf{e}_r \tag{5.433}$$

Therefore,

$$ {}^{\mathcal{F}}\mathbf{a}_{C/O} = -\frac{l\dot{\theta}^2}{2}\mathbf{e}_r + \frac{l\ddot{\theta}}{2}\mathbf{e}_\theta \tag{5.434} $$

which implies that

$$ {}^{\mathcal{F}}\bar{\mathbf{a}} = \ddot{x}\mathbf{E}_x - \frac{l\dot{\theta}^2}{2}\mathbf{e}_r + \frac{l\ddot{\theta}}{2}\mathbf{e}_\theta \tag{5.435} $$

Next, the angular momentum of the rod relative to the center of mass is given as

$$ {}^{\mathcal{F}}\bar{\mathbf{H}} = \bar{\mathbf{I}}^{\mathcal{R}} \cdot {}^{\mathcal{F}}\boldsymbol{\omega}^{\mathcal{R}} \tag{5.436} $$

Now since $\{\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_z\}$ is a principal-axis basis, we have that

$$ \bar{\mathbf{I}}^{\mathcal{R}} = \bar{I}_{rr}\mathbf{e}_r \otimes \mathbf{e}_r + \bar{I}_{\theta\theta}\mathbf{e}_\theta \otimes \mathbf{e}_\theta + \bar{I}_{zz}\mathbf{e}_z \otimes \mathbf{e}_z \tag{5.437} $$

Then, using the expresion for ${}^{\mathcal{F}}\boldsymbol{\omega}^{\mathcal{R}}$ from Eq. (5.421), we obtain ${}^{\mathcal{F}}\bar{\mathbf{H}}$ as

$$ {}^{\mathcal{F}}\bar{\mathbf{H}} = \left(\bar{I}_{rr}\mathbf{e}_r \otimes \mathbf{e}_r + \bar{I}_{\theta\theta}\mathbf{e}_\theta \otimes \mathbf{e}_\theta + \bar{I}_{zz}\mathbf{e}_z \otimes \mathbf{e}_z\right) \cdot \dot{\theta}\mathbf{e}_z = \bar{I}_{zz}\dot{\theta}\mathbf{e}_z \tag{5.438} $$

The rate of change of ${}^{\mathcal{F}}\bar{\mathbf{H}}$ in reference frame $\mathcal{F}$ is then given as

$$ \frac{{}^{\mathcal{F}}d}{dt}\left({}^{\mathcal{F}}\bar{\mathbf{H}}\right) = \bar{I}_{zz}\ddot{\theta}\mathbf{e}_z \tag{5.439} $$

Now for a slender uniform rod we have that $\bar{I}_{zz} = ml^2/12$. Consequently, we have that

$$ \frac{{}^{\mathcal{F}}d}{dt}\left({}^{\mathcal{F}}\bar{\mathbf{H}}\right) == \frac{ml^2}{12}\ddot{\theta}\mathbf{e}_z \tag{5.440} $$

Kinetics

This problem will be solved by applying the following two balance laws: (1) Euler's 1^{st} law and (2) Euler's 2^{nd} law relative to the center of mass of the rod. The free body diagram of the rod is shown in Fig. 5-19.
Using Fig. 5-19, we see that the following three forces act on the rod

$$ \begin{aligned} \mathbf{N} &= \text{Reaction Force of Track} \\ \mathbf{P} &= \text{Known Horizontal Force} \\ m\mathbf{g} &= \text{Force of Gravity} \end{aligned} $$

Now we have that

$$ \mathbf{N} = N\mathbf{E}_y \tag{5.441} $$

$$ \mathbf{P} = P\mathbf{E}_x \tag{5.442} $$

$$ m\mathbf{g} = -mg\mathbf{E}_y \tag{5.443} $$

The resultant force acting on the particle is then given as

$$ \mathbf{F} = \mathbf{N} + \mathbf{P} + m\mathbf{g} = N\mathbf{E}_y + P\mathbf{E}_x - mg\mathbf{E}_y = P\mathbf{E}_x + (N - mg)\mathbf{E}_y \tag{5.444} $$

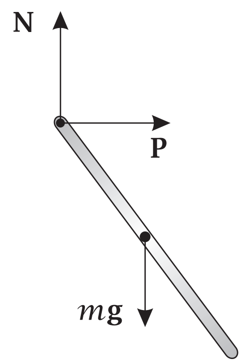

Figure 5-19 Free Body Diagram of Rod for Question 5–11.

Application of Euler's 1^{st} ***Law to Rod***

Then, applying Euler's 1^{st} law, we have that

$$\mathbf{F} = m^{\mathcal{F}}\bar{\mathbf{a}} \tag{5.445}$$

Using the expression for ${}^{\mathcal{F}}\bar{\mathbf{a}}$ from Eq. (5.435) and the resultant force from Eq. (5.444), we obtain

$$P\mathbf{E}_x + (N - mg)\mathbf{E}_y = m\ddot{x}\mathbf{E}_x - \frac{ml\dot{\theta}^2}{2}\mathbf{e}_r + \frac{ml\ddot{\theta}}{2}\mathbf{e}_\theta \tag{5.446}$$

Then, computing the projection of Eq. (5.446) in the $\mathbf{E}_x$-direction, we have that

$$P = m\ddot{x} - \frac{ml\dot{\theta}^2}{2}\mathbf{e}_r \cdot \mathbf{E}_x + \frac{ml\ddot{\theta}}{2}\mathbf{e}_\theta \cdot \mathbf{E}_x \tag{5.447}$$

Now from Eq. (5.413) and Eq. (5.414) we have that

$$\mathbf{e}_r \cdot \mathbf{E}_x = \sin\theta \tag{5.448}$$

$$\mathbf{e}_\theta \cdot \mathbf{E}_x = \cos\theta \tag{5.449}$$

Therefore,

$$P = m\ddot{x} - \frac{ml\dot{\theta}^2}{2}\sin\theta + \frac{ml\ddot{\theta}}{2}\cos\theta \tag{5.450}$$

Next, computing the projection of Eq. (5.446) in the $\mathbf{E}_y$-direction, we have that

$$N - mg = -\frac{ml\dot{\theta}^2}{2}\mathbf{e}_r \cdot \mathbf{E}_y + \frac{ml\ddot{\theta}}{2}\mathbf{e}_\theta \cdot \mathbf{E}_y \tag{5.451}$$

Now from Eq. (5.413) and Eq. (5.414) we have that

$$\mathbf{e}_r \cdot \mathbf{E}_y = -\cos\theta \tag{5.452}$$

$$\mathbf{e}_\theta \cdot \mathbf{E}_y = \sin\theta \tag{5.453}$$

Therefore,

$$N - mg = \frac{ml\dot{\theta}^2}{2}\cos\theta + \frac{ml\ddot{\theta}}{2}\sin\theta \tag{5.454}$$

The two equations that result from the application of Euler's 1^{st} law to the rod are then given from Eq. (5.450) and Eq. (5.454) as

$$P = m\ddot{x} - \frac{ml\dot{\theta}^2}{2}\sin\theta + \frac{ml\ddot{\theta}}{2}\cos\theta \tag{5.455}$$

$$N - mg = \frac{ml\dot{\theta}^2}{2}\cos\theta + \frac{ml\ddot{\theta}}{2}\sin\theta \tag{5.456}$$

Application of Euler's 2^{nd} Law to Rod

Applying Euler's 2^{nd} law to the center of mass of the rod, we have that

$$\bar{\mathbf{M}} = \frac{{}^{\mathcal{F}}d}{dt}\left({}^{\mathcal{F}}\bar{\mathbf{H}}\right) \tag{5.457}$$

Now we already have the rate of change of angular momentum relative to the center of mass of the rod from Eq. (5.439). Next, since gravity passes through the center of mass of the rod and the forces **N** and **P** both act at point O, the moment relative to the center of mass is given as

$$\bar{\mathbf{M}} = (\mathbf{r}_O - \bar{\mathbf{r}}) \times (\mathbf{N} + \mathbf{P}) \tag{5.458}$$

Now we have that

$$\mathbf{r}_O - \bar{\mathbf{r}} = -\frac{l}{2}\mathbf{e}_r \tag{5.459}$$

Furthermore, using the expressions for **N** and **P** from Eq. (5.441) and Eq. (5.442), respectively, we obtain

$$\bar{\mathbf{M}} = -\frac{l}{2}\mathbf{e}_r \times \left(N\mathbf{E}_y + P\mathbf{E}_x\right) = -\frac{l}{2}\left(N\mathbf{e}_r \times \mathbf{E}_y + P\mathbf{e}_r \times \mathbf{E}_x\right) \tag{5.460}$$

Now we have that

$$\mathbf{e}_r \times \mathbf{E}_y = \sin\theta\,\mathbf{E}_z = \sin\theta\,\mathbf{e}_z \tag{5.461}$$

$$\mathbf{e}_r \times \mathbf{E}_x = \cos\theta\,\mathbf{E}_z = \cos\theta\,\mathbf{e}_z \tag{5.462}$$

Consequently,

$$\bar{\mathbf{M}} = -\frac{l}{2}\mathbf{e}_r \times \left(N\mathbf{E}_y + P\mathbf{E}_x\right) = -\frac{l}{2}\left(N\sin\theta + P\cos\theta\right)\mathbf{e}_z \tag{5.463}$$

Then, setting $\bar{\mathbf{M}}$ in Eq. (5.463) equal to ${}^{\mathcal{F}}d({}^{\mathcal{F}}\bar{\mathbf{H}})/dt$ using the expression for ${}^{\mathcal{F}}d({}^{\mathcal{F}}\bar{\mathbf{H}})/dt$ from Eq. (5.440), we obtain

$$-\frac{l}{2}\left(N\sin\theta + P\cos\theta\right)\mathbf{e}_z = \frac{ml^2}{12}\ddot{\theta}\mathbf{e}_z \tag{5.464}$$

Eq. (5.464) simplifies to

$$N \sin\theta + P\cos\theta = -\frac{ml}{6}\ddot{\theta} \tag{5.465}$$

Determination of System of Two Differential Equations of Motion

The first differential equation of motion is obtained directly from Eq. (5.455), i.e.,

$$m\ddot{x} + \frac{ml\ddot{\theta}}{2}\cos\theta - \frac{ml\dot{\theta}^2}{2}\sin\theta = P \tag{5.466}$$

The second differential equation is obtained using Eq. (5.456) and Eq. (5.465). First, multiplying Eq. (5.456) by $\sin\theta$, we have that

$$N\sin\theta - mg\sin\theta = \frac{ml\dot{\theta}^2}{2}\cos\theta\sin\theta + \frac{ml\ddot{\theta}}{2}\sin^2\theta \tag{5.467}$$

Then, subtracting Eq. (5.465) from Eq. (5.467), we obtain

$$P\cos\theta + mg\sin\theta = -\frac{ml}{6}\ddot{\theta} - \frac{ml\dot{\theta}^2}{2}\cos\theta\sin\theta - \frac{ml\ddot{\theta}}{2}\sin^2\theta \tag{5.468}$$

Simplifying Eq. (5.467), we obtain the second differential equation of motion as

$$\frac{ml}{6}\left(1 + 3\sin^2\theta\right)\ddot{\theta} + \frac{ml\dot{\theta}^2}{2}\cos\theta\sin\theta + mg\sin\theta = -P\cos\theta \tag{5.469}$$

A system of two differential describing the motion of the rod is then given as

$$\begin{aligned} m\ddot{x} + \frac{ml\ddot{\theta}}{2}\cos\theta - \frac{ml\dot{\theta}^2}{2}\sin\theta &= P && (5.470)\\ \frac{ml}{6}\left(1 + 3\sin^2\theta\right)\ddot{\theta} + \frac{ml\dot{\theta}^2}{2}\cos\theta\sin\theta + mg\sin\theta &= -P\cos\theta && (5.471)\end{aligned}$$

Alternate System of Two Differential Equations

Now while Eqs. (5.470) and (5.471) are perfectly valid, a more elegant system of differential equations is obtained by manipulating Eqs. (5.470) and (5.471). First, multiplying Eq. (5.470) by $\cos\theta$, we obtain

$$m\ddot{x}\cos\theta + \frac{ml\ddot{\theta}}{2}\cos^2\theta - \frac{ml\dot{\theta}^2}{2}\sin\theta\cos\theta = P\cos\theta \tag{5.472}$$

Then, adding Eqs. (5.472) and (5.471) gives

$$\begin{aligned} &m\ddot{x}\cos\theta + \frac{ml\ddot{\theta}}{2}\cos^2\theta - \frac{ml\dot{\theta}^2}{2}\sin\theta\cos\theta + \frac{ml}{6}\left(1 + 3\sin^2\theta\right)\ddot{\theta}\\ &+ \frac{ml\dot{\theta}^2}{2}\cos\theta\sin\theta + mg\sin\theta = 0 \end{aligned} \tag{5.473}$$

Now it is seen that the second and fourth terms in Eq. (5.473) cancel. Consequently, Eq. (5.473) simplifies to

$$m\ddot{x}\cos\theta + \frac{ml\ddot{\theta}}{2}\cos^2\theta + \frac{ml}{6}\left(1 + 3\sin^2\theta\right)\ddot{\theta} + mg\sin\theta = 0 \tag{5.474}$$

Eq. (5.474) can be further simplified to give

$$m\ddot{x}\cos\theta + \frac{2ml}{3}\ddot{\theta} + mg\sin\theta = 0 \tag{5.475}$$

An alternate system of differential equations is then obtained using Eq. (5.470) and (5.475) as

$$m\ddot{x} + \frac{ml\ddot{\theta}}{2}\cos\theta - \frac{ml\dot{\theta}^2}{2}\sin\theta = P \tag{5.476}$$

$$m\ddot{x}\cos\theta + \frac{2ml\ddot{\theta}}{3} + mg\sin\theta = 0 \tag{5.477}$$

Derivation of 2^{nd} Differential Equation Using Point O As Reference Point

It is ntoed that one of the differential equations can be obtained by using point O as the reference point. In particular, we know from Eq. (5.430 that the acceleration of point O in reference frame $\mathcal{F}$ is ${}^{\mathcal{F}}\mathbf{a}_O = \ddot{x}\mathbf{E}_x \neq \mathbf{0}$. Consequently, point O is not inertially fixed and it is necessary to apply the general form of Euler's 2^{nd} law relative to point O, i.e.,

$$\mathbf{M}_O - (\bar{\mathbf{r}} - \mathbf{r}_O) \times m{}^{\mathcal{F}}\mathbf{a}_O = \frac{{}^{\mathcal{F}}d}{dt}\left({}^{\mathcal{F}}\mathbf{H}_O\right) \tag{5.478}$$

Now examining the free body diagram of Fig. (5-19), we see that the forces **P** and **N** pass through point O. Consequently, the moment relative to point O is due entirely to the force of gravity and is given as

$$\mathbf{M}_O = (\bar{\mathbf{r}} - \mathbf{r}_O) \times m\mathbf{g} \tag{5.479}$$

Using the expressions for $\mathbf{r}_O - \bar{\mathbf{r}}$ from Eq. (5.459), we have that

$$\bar{\mathbf{r}} - \mathbf{r}_O = \frac{l}{2}\mathbf{e}_r \tag{5.480}$$

Next the force of gravity is given as

$$m\mathbf{g} = -mg\mathbf{E}_y = -mg(-\cos\theta\,\mathbf{e}_r + \sin\theta\,\mathbf{e}_\theta) = mg\cos\theta\,\mathbf{e}_r - mg\sin\theta\,\mathbf{e}_\theta \tag{5.481}$$

where we have used the expression for $\mathbf{E}_y$ from Eq. (5.416). The moment relative to point O is then given as

$$\mathbf{M}_O = \frac{l}{2}\mathbf{e}_r \times (mg\cos\theta\,\mathbf{e}_r - mg\sin\theta\,\mathbf{e}_\theta) = -\frac{mgl}{2}\sin\theta\,\mathbf{e}_z \tag{5.482}$$

Next, the inertial moment due to the acceleration of point O is then given as

$$-(\bar{\mathbf{r}} - \mathbf{r}_O) \times m^{\mathcal{F}}\mathbf{a}_O = \frac{l}{2}\mathbf{e}_r \times m\ddot{x}\mathbf{E}_x \tag{5.483}$$

Using the expression for $\mathbf{E}_x$ from Eq. (5.415), we obtain

$$-(\bar{\mathbf{r}} - \mathbf{r}_O) \times m^{\mathcal{F}}\mathbf{a}_O = -\frac{l}{2}\mathbf{e}_r \times m\ddot{x}(\sin\theta\,\mathbf{e}_r + \cos\theta\,\mathbf{e}_\theta) = -\frac{ml\ddot{x}}{2}\cos\theta\,\mathbf{e}_z \tag{5.484}$$

Then, the angular momentum relative to point O is obtained as

$${}^{\mathcal{F}}\mathbf{H}_O = {}^{\mathcal{F}}\bar{\mathbf{H}} + (\mathbf{r}_O - \bar{\mathbf{r}}) \times m({}^{\mathcal{F}}\mathbf{v}_O - {}^{\mathcal{F}}\bar{\mathbf{v}}) = \frac{ml^2}{12}\dot{\theta}\mathbf{e}_z + \left(-\frac{l}{2}\mathbf{e}_r\right) m \left(\frac{l\dot{\theta}}{2}\mathbf{e}_\theta\right) \tag{5.485}$$

Simplifying Eq. (5.485) gives

$${}^{\mathcal{F}}\mathbf{H}_O = \frac{ml^2}{12}\dot{\theta}\mathbf{e}_z + \frac{ml^2}{4}\dot{\theta}\mathbf{e}_z = \frac{ml^2}{3}\dot{\theta}\mathbf{e}_z \tag{5.486}$$

The rate of change of ${}^{\mathcal{F}}\mathbf{H}_O$ in reference frame $\mathcal{F}$ is then given as

$$\frac{{}^{\mathcal{F}}d}{dt}\left({}^{\mathcal{F}}\mathbf{H}_O\right) = \frac{ml^2}{3}\ddot{\theta}\mathbf{e}_z \tag{5.487}$$

Substituting the results of Eqs. (5.482), (5.484), and (5.487) into (5.478), we obtain

$$-\frac{mgl}{2}\sin\theta\,\mathbf{e}_z - \frac{ml\ddot{x}}{2}\cos\theta\,\mathbf{e}_z = \frac{ml^2}{3}\ddot{\theta}\mathbf{e}_z \tag{5.488}$$

Dropping the dependence in Eq. (5.488) and simplifying, we obtain

$$m\ddot{x}\cos\theta + \frac{2ml}{3}\ddot{\theta} + mg\sin\theta = 0 \tag{5.489}$$

It is seen that Eq. (5.489) is identical to Eq. (5.475).

Question 5–12

A uniform slender rod of mass m and length $2l$ slides without friction along a fixed circular track of radius R as shown in Fig. P5-12. Knowing that θ is the angle from the vertical to the center of the rod and that gravity acts downward, determine the differential equation of motion for the rod.

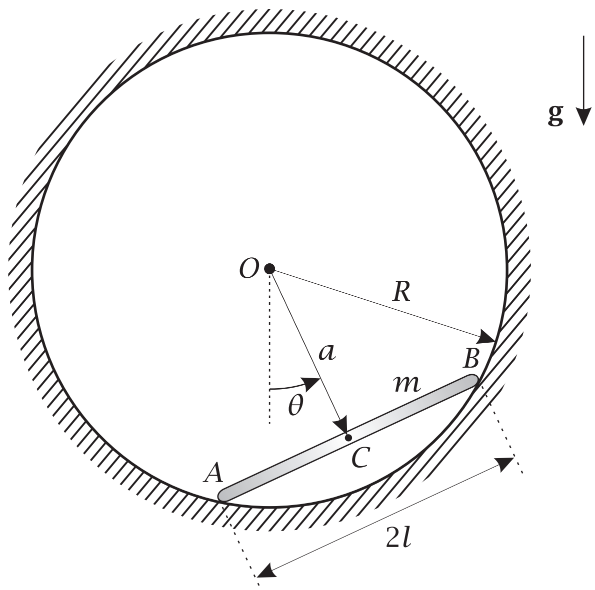

Figure P5-18

Solution to Question 5–12

Kinematics

First, let $\mathcal{F}$ be a reference frame fixed to the track. Then, choose the following coordinate system fixed in reference frame $\mathcal{F}$:

$$
\begin{array}{ccr}
 & \text{Origin at } O & \\
\mathbf{E}_x & = & \text{along } OC \text{ When } \theta = 0 \\
\mathbf{E}_z & = & \text{Out of Page} \\
\mathbf{E}_y & = & \mathbf{E}_z \times \mathbf{E}_x
\end{array}
$$

Next, let $\mathcal{R}$ be a reference frame fixed to the rod. Then, choose the following coordinate system fixed in reference frame $\mathcal{R}$:

$$
\begin{array}{ccr}
 & \text{Origin at } O & \\
\mathbf{e}_r & = & \text{Along } OC \\
\mathbf{e}_z & = & \text{Out of Page} \\
\mathbf{e}_\theta & = & \mathbf{e}_z \times \mathbf{e}_r
\end{array}
$$

The geometry of the coordinate systems $\{\mathbf{E}_x, \mathbf{E}_y, \mathbf{E}_z\}$ and $\{\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_z\}$ is shown in Fig. 5-20.

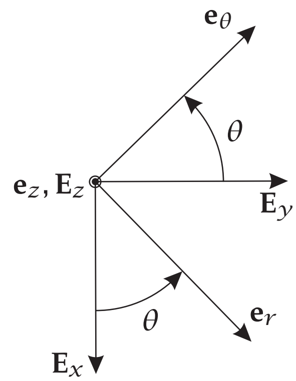

Figure 5-20 Unit Vertical Direction for Question 5–12.

Using Fig. 5-20, we have that

$$\begin{aligned} \mathbf{E}_x &= \cos\theta\,\mathbf{e}_r - \sin\theta\,\mathbf{e}_\theta && (5.490) \\ \mathbf{E}_y &= \sin\theta\,\mathbf{e}_r + \cos\theta\,\mathbf{e}_\theta && (5.491) \end{aligned}$$

Next, the position of the center of mass of the rod is given as

$$\bar{\mathbf{r}} = \mathbf{r}_C = a\mathbf{e}_r \tag{5.492}$$

Furthermore, the angular velocity of reference frame $\mathcal{R}$ in reference frame $\mathcal{F}$ is given as

$$^{\mathcal{F}}\boldsymbol{\omega}^{\mathcal{R}} = \dot{\theta}\mathbf{E}_z \tag{5.493}$$

Then, applying the rate of change transport theorem between reference frame $\mathcal{R}$ and reference frame $\mathcal{F}$, we obtain the velocity of the center of mass of the rod as

$$^{\mathcal{F}}\bar{\mathbf{v}} = \frac{^{\mathcal{F}}d\bar{\mathbf{r}}}{dt} = \frac{^{\mathcal{R}}d\bar{\mathbf{r}}}{dt} + {}^{\mathcal{F}}\boldsymbol{\omega}^{\mathcal{R}} \times \bar{\mathbf{r}} \tag{5.494}$$

Now we have that

$$\begin{aligned} \frac{^{\mathcal{R}}d\bar{\mathbf{r}}}{dt} &= \mathbf{0} && (5.495) \\ {}^{\mathcal{F}}\boldsymbol{\omega}^{\mathcal{R}} \times \bar{\mathbf{r}} &= \dot{\theta}\mathbf{e}_z \times (a\mathbf{e}_r) = a\dot{\theta}\mathbf{e}_\theta && (5.496) \end{aligned}$$

Adding Eq. (5.495) and Eq. (5.496), we obtain the velocity of the center of mass of the rod in reference frame $\mathcal{F}$ as Differentiating $\mathbf{v}_C$ in Eq. (5.497), we obtain

$$^{\mathcal{F}}\bar{\mathbf{v}} = a\dot{\theta}\mathbf{e}_\theta \tag{5.497}$$

Then, applying the rate of change transport theorem to ${}^{\mathcal{F}}\bar{\mathbf{v}}$, we obtain the acceleration of the center of mass of the rod in reference frame $\mathcal{F}$ as

$$ {}^{\mathcal{F}}\bar{\mathbf{a}} = \frac{{}^{\mathcal{F}}d}{dt}\left({}^{\mathcal{F}}\bar{\mathbf{a}}\right) = \frac{{}^{\mathcal{R}}d}{dt}\left({}^{\mathcal{F}}\bar{\mathbf{v}}\right) + {}^{\mathcal{F}}\boldsymbol{\omega}^{\mathcal{R}} \times {}^{\mathcal{F}}\bar{\mathbf{v}} \tag{5.498} $$

Now we have that

$$ \frac{{}^{\mathcal{R}}d}{dt}\left({}^{\mathcal{F}}\bar{\mathbf{v}}\right) = a\ddot{\theta}\mathbf{e}_\theta \tag{5.499} $$

$$ {}^{\mathcal{F}}\boldsymbol{\omega}^{\mathcal{R}} \times {}^{\mathcal{F}}\bar{\mathbf{v}} = \dot{\theta}\mathbf{e}_z \times (a\dot{\theta}\mathbf{e}_\theta) = -a\dot{\theta}^2\mathbf{e}_r \tag{5.500} $$

Adding Eq. (5.499) and Eq. (5.500), we obtain the acceleration of the center of mass of the rod in reference frame $\mathcal{F}$ as

$$ {}^{\mathcal{F}}\bar{\mathbf{a}} = -a\dot{\theta}^2\mathbf{e}_r + a\ddot{\theta}\mathbf{e}_\theta \tag{5.501} $$

Next, from the kinematic properties of a rigid body, we have that

$$ {}^{\mathcal{F}}\mathbf{v}_A = {}^{\mathcal{F}}\bar{\mathbf{v}} + {}^{\mathcal{F}}\boldsymbol{\omega}^{\mathcal{R}}(\mathbf{r}_A - \mathbf{r}_C) \tag{5.502} $$

$$ {}^{\mathcal{F}}\mathbf{v}_B = {}^{\mathcal{F}}\bar{\mathbf{v}} + {}^{\mathcal{F}}\boldsymbol{\omega}^{\mathcal{R}}(\mathbf{r}_B - \mathbf{r}_C) \tag{5.503} $$

Now we note that

$$ \mathbf{r}_A - \mathbf{r}_C = -l\mathbf{e}_\theta \tag{5.504} $$

$$ \mathbf{r}_B - \mathbf{r}_C = l\mathbf{e}_\theta \tag{5.505} $$

Substituting $\mathbf{r}_A - \mathbf{r}_C$ and $\mathbf{r}_B - \mathbf{r}_C$ into Eq. (5.502) and Eq. (5.503), respectively, we obtain

$$ \mathbf{v}_A = a\dot{\theta}\mathbf{e}_\theta + \dot{\theta}\mathbf{E}_z \times (-l\mathbf{e}_\theta) = l\dot{\theta}\mathbf{e}_r + a\dot{\theta}\mathbf{e}_\theta \tag{5.506} $$

$$ \mathbf{v}_B = a\dot{\theta}\mathbf{e}_\theta + \dot{\theta}\mathbf{E}_z \times (l\mathbf{e}_\theta) = -l\dot{\theta}\mathbf{e}_r + a\dot{\theta}\mathbf{e}_\theta \tag{5.507} $$

Kinetics

The free body diagram of the rod is shown in Fig. 5-21. It can be seen that the following three forces act on the rod:

$\mathbf{N}_A$ = Reaction Force of Track at Point A
$\mathbf{N}_B$ = Reaction Force of Track at Point B
$m\mathbf{g}$ = Force of Gravity

From the geometry of the problem, it is seen that the reaction forces $\mathbf{N}_A$ and $\mathbf{N}_B$ are in the directions orthogonal to the track at points A and B, respectively. Suppose we let $\mathbf{u}_A$ and $\mathbf{u}_B$ be the directions of $\mathbf{N}_A$ and $\mathbf{N}_B$, respectively. Then we can write

$$ \mathbf{N}_A = N_A\mathbf{u}_A \tag{5.508} $$

$$ \mathbf{N}_B = N_B\mathbf{u}_B \tag{5.509} $$

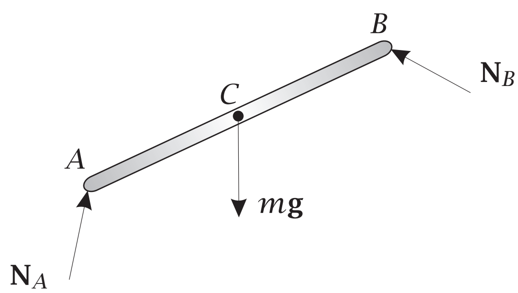

Figure 5-21 Free Body Diagram for Question 4.

Now, it can be seen that $\mathbf{u}_A$ and $\mathbf{u}_B$ must lie along the line segments from O to A and O to B, respectively. Therefore,

$$\mathbf{u}_A = \mathbf{r}_A / \|\mathbf{r}_A\| \tag{5.510}$$

$$\mathbf{u}_B = \mathbf{r}_B / \|\mathbf{r}_B\| \tag{5.511}$$

Noting that $\mathbf{r}_A = a\mathbf{e}_r - l\mathbf{e}_\theta$ and $\mathbf{r}_B = a\mathbf{e}_r + l\mathbf{e}_\theta$, we obtain

$$\mathbf{u}_A = \frac{a\mathbf{e}_r - l\mathbf{e}_\theta}{\sqrt{a^2 + l^2}} \tag{5.512}$$

$$\mathbf{u}_B = \frac{a\mathbf{e}_r + l\mathbf{e}_\theta}{\sqrt{a^2 + l^2}} \tag{5.513}$$

which gives

$$\mathbf{N}_A = N_A \frac{a\mathbf{e}_r - l\mathbf{e}_\theta}{\sqrt{a^2 + l^2}} \tag{5.514}$$

$$\mathbf{N}_B = N_B \frac{a\mathbf{e}_r + l\mathbf{e}_\theta}{\sqrt{a^2 + l^2}} \tag{5.515}$$

Next, the force of gravity is given as

$$m\mathbf{g} = mg\mathbf{E}_x \tag{5.516}$$

Using the expression for $\mathbf{E}_x$ from Eq. (5.490), the force of gravity is obtained as

$$m\mathbf{g} = mg\mathbf{E}_x = mg(\cos\theta\,\mathbf{e}_r - \sin\theta\,\mathbf{e}_\theta) = mg\cos\theta\,\mathbf{e}_r - mg\sin\theta\,\mathbf{e}_\theta \tag{5.517}$$

Now we will solve this problem by the following two methods:

- Euler's Laws Relative to the Center of Mass of the Rod
- The Work-Energy Theorem for a Rigid Body

Method 1: Euler's Laws Using Center of Mass of Rod as Reference Point

Application of Euler's 1^{st} ***Law to Rod***

Using the free body diagram of Fig. 5-21, the resultant force acting on the rod is given as

$$\mathbf{F} = \mathbf{N}_A + \mathbf{N}_B + m\mathbf{g} \tag{5.518}$$

Substituting $\mathbf{N}_A$, $\mathbf{N}_B$, and $m\mathbf{g}$ from Eq. (5.514), Eq. (5.515), and Eq. (5.517) and $\mathbf{a}_C$ from Eq. (5.501), we obtain

$$\mathbf{F} = N_A \frac{a\mathbf{e}_r - l\mathbf{e}_\theta}{\sqrt{a^2 + l^2}} + N_B \frac{a\mathbf{e}_r + l\mathbf{e}_\theta}{\sqrt{a^2 + l^2}} + mg\cos\theta\,\mathbf{e}_r - mg\sin\theta\,\mathbf{e}_\theta \tag{5.519}$$

Then, setting $\mathbf{F}$ equal to $m^{\mathcal{F}}\bar{\mathbf{a}}$ using the expression for ${}^{\mathcal{F}}\bar{\mathbf{a}}$ from Eq. (5.501), we obtain

$$\mathbf{F} = N_A \frac{a\mathbf{e}_r - l\mathbf{e}_\theta}{\sqrt{a^2 + l^2}} + N_B \frac{a\mathbf{e}_r + l\mathbf{e}_\theta}{\sqrt{a^2 + l^2}} + mg\cos\theta\,\mathbf{e}_r - mg\sin\theta\,\mathbf{e}_\theta = -ma\dot{\theta}^2\mathbf{e}_r + ma\ddot{\theta}\mathbf{e}_\theta \tag{5.520}$$

Rearranging Eq. (5.520) gives

$$\left[\frac{a}{\sqrt{a^2 + l^2}}(N_A + N_B) + mg\cos\theta\right]\mathbf{e}_r - \left[\frac{l}{\sqrt{a^2 + l^2}}(N_A - N_B) + mg\sin\theta\right] = -ma\dot{\theta}^2\mathbf{e}_r + ma\ddot{\theta}\mathbf{e}_\theta \tag{5.521}$$

Equating components, we obtain the following two scalar equations:

$$-ma\dot{\theta}^2 = \frac{a}{\sqrt{a^2 + l^2}}(N_A + N_B) + mg\cos\theta \tag{5.522}$$

$$ma\ddot{\theta} = -\frac{l}{\sqrt{a^2 + l^2}}(N_A - N_B) - mg\sin\theta \tag{5.523}$$

Application of Euler's 2^{nd} ***Law Relative to Center of Mass ofRod***

Observing that $m\mathbf{g}$ passes through the center of mass, the moment relative to the center of mass of the rod is given as

$$\bar{\mathbf{M}} = \mathbf{M}_C = (\mathbf{r}_A - \mathbf{r}_C) \times \mathbf{N}_A + (\mathbf{r}_B - \mathbf{r}_C) \times \mathbf{N}_B \tag{5.524}$$

Substituting the expressions for $\mathbf{r}_A - \mathbf{r}_C$ and $\mathbf{r}_B - \mathbf{r}_C$ from Eq. (5.504 and Eq. (5.505, respectively, and the expressions for $\mathbf{N}_A$ and $\mathbf{N}_B$ from Eq. (5.514) and Eq. (5.515), respectively, into Eq. (5.524), we obtain $\bar{\mathbf{M}}$ as

$$\bar{\mathbf{M}} = (-l\mathbf{e}_\theta) \times N_A \frac{a\mathbf{e}_r - l\mathbf{e}_\theta}{\sqrt{a^2 + l^2}} + (l\mathbf{e}_\theta) \times N_B \frac{a\mathbf{e}_r + l\mathbf{e}_\theta}{\sqrt{a^2 + l^2}} \tag{5.525}$$

Eq. (5.525) simplifies to

$$\bar{\mathbf{M}} = \frac{al}{\sqrt{a^2 + l^2}}(N_A - N_B)\,\mathbf{e}_z \tag{5.526}$$

Next, the angular momentum of the rod relative to the center of mass of the rod is given as

$$ {}^{\mathcal{F}}\bar{\mathbf{H}} = \bar{\mathbf{I}}^{\mathcal{R}} \cdot {}^{\mathcal{F}}\boldsymbol{\omega}^{\mathcal{R}} \tag{5.527} $$

Now since $\{\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_z\}$ is a principal-axis basis, we have that

$$ \bar{\mathbf{I}}^{\mathcal{R}} = \bar{I}_{rr}\mathbf{e}_r \otimes \mathbf{e}_r + \bar{I}_{\theta\theta}\mathbf{e}_\theta \otimes \mathbf{e}_\theta + \bar{I}_{zz}\mathbf{e}_z \otimes \mathbf{e}_z \tag{5.528} $$

Consequently,

$$ {}^{\mathcal{F}}\bar{\mathbf{H}} = \bar{I}_{zz}\dot{\theta}\mathbf{e}_z \tag{5.529} $$

Now for a slender uniform rod we have that

$$ \bar{I}_{zz} = \frac{mL^2}{12} = \frac{m(2l)^2}{12} = \frac{ml^2}{3} \tag{5.530} $$

Where $L = 2l$ is the length of the rod. Therefore, we obtain ${}^{\mathcal{F}}\bar{\mathbf{H}}$ as

$$ {}^{\mathcal{F}}\bar{\mathbf{H}} = \frac{ml^2}{3}\dot{\theta}\mathbf{e}_z \tag{5.531} $$

The rate of change of ${}^{\mathcal{F}}\bar{\mathbf{H}}$ in reference frame $\mathcal{F}$ is then obtained as

$$ \frac{{}^{\mathcal{F}}d}{dt}\left({}^{\mathcal{F}}\bar{\mathbf{H}}\right) = \frac{ml^2}{3}\ddot{\theta}\mathbf{e}_z \tag{5.532} $$

Setting $\bar{\mathbf{M}}$ from Eq. (5.526) equal to $\dot{\bar{\mathbf{H}}}$ from Eq. (5.532), we obtain

$$ \frac{ml^2}{3}\ddot{\theta} = \frac{al}{\sqrt{a^2 + l^2}}(N_A - N_B) \tag{5.533} $$

The differential equation of motion is obtained using Eq. (5.523) and Eq. (5.533). Dividing Eq. (5.523) by a gives

$$ \frac{ml^2}{3a}\ddot{\theta} = \frac{l}{\sqrt{a^2 + l^2}}(N_A - N_B) \tag{5.534} $$

Then, adding Eq. (5.533) and Eq. (5.534), we obtain

$$ ma\ddot{\theta} + \frac{ml^2}{3a}\ddot{\theta} = -mg\sin\theta \tag{5.535} $$

Rearranging Eq. (5.535), we obtain the differential equation of motion as

$$ (3ma^2 + ml^2)\ddot{\theta} + 3mga\sin\theta = 0 \tag{5.536} $$

Method 2: Work-Energy Theorem

From the work-energy theorem for a rigid body we have that

$$\frac{d}{dt}\left({}^{\mathcal{F}}E\right) = \sum_{i=1}^{n} \mathbf{F}_i^{nc} \cdot {}^{\mathcal{F}}\mathbf{v}_i + \boldsymbol{\tau}^{nc} \cdot {}^{\mathcal{F}}\boldsymbol{\omega}^{\mathcal{R}} \tag{5.537}$$

Now since the only non-conservative forces acting on the rod are $\mathbf{N}_A$ and $\mathbf{N}_B$, the first term in Eq. (5.537) is given as

$$\sum_{i=1}^{n} \mathbf{F}_i^{nc} \cdot {}^{\mathcal{F}}\mathbf{v}_i = \mathbf{N}_A \cdot {}^{\mathcal{F}}\mathbf{v}_A + \mathbf{N}_B \cdot {}^{\mathcal{F}}\mathbf{v}_B \tag{5.538}$$

Using
$lsup\mathcal{F}\mathbf{v}_A$ and ${}^{\mathcal{F}}\mathbf{v}_B$ from Eq. (5.506) and Eq. (5.507), respectively, and $\mathbf{N}_A$ and $\mathbf{N}_B$ from Eq. (5.514) and Eq. (5.515), respectively, we obtain

$$\sum_{i=1}^{n} \mathbf{F}_i^{nc} \cdot {}^{\mathcal{F}}\mathbf{v}_i = N_A \frac{a\mathbf{e}_r - l\mathbf{e}_\theta}{\sqrt{a^2+l^2}} \cdot (l\dot{\theta}\mathbf{e}_r + a\dot{\theta}\mathbf{e}_\theta) + N_B \frac{a\mathbf{e}_r + l\mathbf{e}_\theta}{\sqrt{a^2+l^2}} \cdot (l\dot{\theta}\mathbf{e}_r - a\dot{\theta}\mathbf{e}_\theta) = 0 \tag{5.539}$$

Furthermore, since no pure torques act on the rod, we have that

$$\boldsymbol{\tau}^{nc} \cdot {}^{\mathcal{F}}\boldsymbol{\omega}^{\mathcal{R}} = 0 \tag{5.540}$$

Consequently,

$$\frac{d}{dt}\left({}^{\mathcal{F}}E\right) = 0 \tag{5.541}$$

which implies that the total energy is a constant, i.e.

$${}^{\mathcal{F}}E = {}^{\mathcal{F}}T + {}^{\mathcal{F}}U = \text{constant} \tag{5.542}$$

Now the kinetic energy in reference frame $\mathcal{F}$ is given as

$${}^{\mathcal{F}}T = \frac{1}{2}m{}^{\mathcal{F}}\bar{\mathbf{v}} \cdot {}^{\mathcal{F}}\bar{\mathbf{v}} + \frac{1}{2}{}^{\mathcal{F}}\bar{\mathbf{H}} \cdot {}^{\mathcal{F}}\boldsymbol{\omega}^{\mathcal{R}} \tag{5.543}$$

Substituting ${}^{\mathcal{F}}\bar{\mathbf{v}}$ from Eq. (5.497) and ${}^{\mathcal{F}}\bar{\mathbf{H}}$ from Eq. (5.527), we obtain

$${}^{\mathcal{F}}T = \frac{1}{2}(a\dot{\theta}\mathbf{e}_\theta) \cdot (a\dot{\theta}\mathbf{e}_\theta) + \frac{1}{2}\left(\frac{ml^2}{3}\dot{\theta}\mathbf{E}_z\right) \cdot \dot{\theta}\mathbf{E}_z = \frac{3ma^2 + ml^2}{6}\dot{\theta}^2 \tag{5.544}$$

Next, the potential energy is due entirely to gravity and is given as

$${}^{\mathcal{F}}U = -m\mathbf{g} \cdot \bar{\mathbf{r}} \tag{5.545}$$

Using the expression for $m\mathbf{g}$ from Eq. (5.517) and $\bar{\mathbf{r}} = \mathbf{r}_C$ from Eq. (5.492), we obtain

$${}^{\mathcal{F}}U = -(mg\cos\theta\,\mathbf{e}_r - mg\sin\theta\,\mathbf{e}_\theta) \cdot (a\mathbf{e}_r) = -mga\cos\theta \tag{5.546}$$

Adding the kinetic energy in Eq. (5.544), and the potential energy in Eq. (5.546), we obtain

$$^{\mathcal{F}}E = \frac{3ma^2 + ml^2}{6}\dot{\theta}^2 - mga\cos\theta = \text{constant} \tag{5.547}$$

Computing the rate of change of $^{\mathcal{F}}E$ in Eq. (5.547), we obtain

$$\frac{d}{dt}\left(^{\mathcal{F}}E\right) = \frac{3ma^2 + ml^2}{3}\dot{\theta}\ddot{\theta} + mga\dot{\theta}\sin\theta = 0 \tag{5.548}$$

Noting that $\dot{\theta} \neq 0$ as a function of time, we obtain

$$\frac{3ma^2 + ml^2}{3}\ddot{\theta} + mga\sin\theta = 0 \tag{5.549}$$

Multiplying this last equation through by three, we obtain

$$(3ma^2 + ml^2)\ddot{\theta} + 3mga\sin\theta = 0 \tag{5.550}$$

Solution to Question 5–16

A homogeneous sphere of mass m and radius r rolls without slip along a horizontal surface as shown in Fig. P5-16. The variable x describes the position of the center of mass of the sphere. A horizontal force $\mathbf{P}$ is then applied at a height h from the surface such that $\mathbf{P}$ lies in the vertical plane that contains the center of mass of the sphere. Knowing that gravity acts downward, determine the differential equation of motion for the sphere in terms of the variable x.

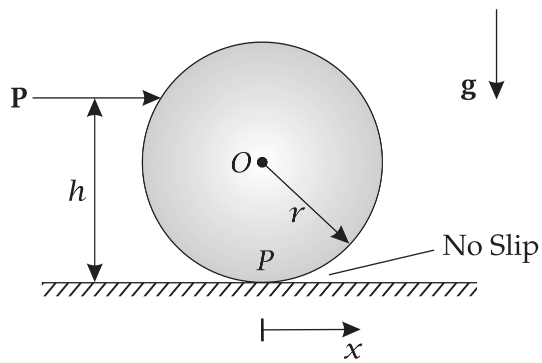

Figure P5-21

Solution to Question 5–16

Kinematics

Let $\mathcal{G}$ be the ground. Then choose the following coordinate system fixed in reference frame $\mathcal{G}$:

	Origin at O When $x = 0$	
$\mathbf{E}_x$	=	To The Right
$\mathbf{E}_z$	=	Into Page
$\mathbf{E}_y$	=	$\mathbf{E}_z \times \mathbf{E}_x$

Next, let S be the sphere. Then choose the following coordinate system fixed in S:

	Origin at O	
$\mathbf{e}_r$	=	Fixed to Sphere
$\mathbf{e}_z$	=	$\mathbf{E}_z$
$\mathbf{e}_\theta$	=	$\mathbf{e}_z \times \mathbf{e}_r$

The position of the center of mass of the sphere is then given as

$$\bar{\mathbf{r}} = \mathbf{r}_O = x\mathbf{E}_x \tag{5.551}$$

Therefore,

$$^{\mathcal{G}}\mathbf{v}_O = \dot{x}\mathbf{E}_x \tag{5.552}$$

Now because the sphere rolls without slip along the ground, we have

$$^{\mathcal{G}}\mathbf{v}_O = {}^{\mathcal{G}}\mathbf{v}_P{}^{\mathcal{S}} = \mathbf{0}. \tag{5.553}$$

Furthermore.

$$^{\mathcal{G}}\mathbf{v}_O = {}^{\mathcal{G}}\mathbf{v}_P + {}^{\mathcal{G}}\boldsymbol{\omega}^{\mathcal{S}} \times (\mathbf{r}_O - \mathbf{r}_P) \tag{5.554}$$

Therefore,

$$^{\mathcal{G}}\mathbf{v}_O = {}^{\mathcal{G}}\boldsymbol{\omega}^{\mathcal{S}} \times (\mathbf{r}_O - \mathbf{r}_P) = \omega\mathbf{e}_z \times (-r\mathbf{E}_y) = r\omega\mathbf{E}_x. \tag{5.555}$$

We then obtain

$$\omega = \frac{\dot{x}}{r} \tag{5.556}$$

which implies that

$$^{\mathcal{G}}\boldsymbol{\omega}^{\mathcal{S}} = \frac{\dot{x}}{r}\mathbf{e}_z. \tag{5.557}$$

Finally, the acceleration of the center of mass of the sphere is given as

$$^{\mathcal{G}}\mathbf{a}_O = \ddot{x}\mathbf{E}_x. \tag{5.558}$$

Next, the angular momentum of the sphere relative to the center of mass of the sphere is given as

$$^{\mathcal{G}}\bar{\mathbf{H}} = \bar{\mathbf{I}}^{\mathcal{S}} \cdot {}^{\mathcal{G}}\boldsymbol{\omega}^{\mathcal{S}} \tag{5.559}$$

Because $\{\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_z\}$ is a principal-axis basis, we have

$$\bar{\mathbf{I}}^{\mathcal{S}} = \bar{I}_{rr}\mathbf{e}_r \otimes \mathbf{e}_r + \bar{I}_{\theta\theta}\mathbf{e}_\theta \otimes \mathbf{e}_\theta + \bar{I}_{zz}\mathbf{e}_z \otimes \mathbf{e}_z \tag{5.560}$$

Using the fact that ${}^{\mathcal{G}}\boldsymbol{\omega}^{\mathcal{S}} = \omega\mathbf{e}_z$, we obtain

$$^{\mathcal{G}}\bar{\mathbf{H}} = \bar{I}_{zz}\omega\mathbf{e}_z = \frac{2}{5}mr^2\frac{\dot{x}}{r}\mathbf{e}_z = \frac{2}{5}mr\dot{x}\mathbf{e}_z. \tag{5.561}$$

The rate of change of ${}^{\mathcal{G}}\bar{\mathbf{H}}$ in reference frame $\mathcal{G}$ is then given as

$$^{\mathcal{G}}\frac{d}{dt}\left({}^{\mathcal{G}}\bar{\mathbf{H}}\right) = \frac{2}{5}mr\ddot{x}\mathbf{e}_z. \tag{5.562}$$

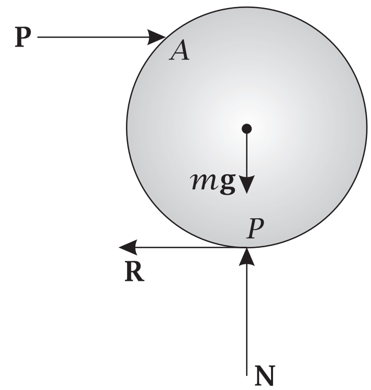

Figure 5-22 Free Body Diagram of Rod for Question 5–16.

Kinetics

This problem will be solved by applying the following two balance laws: (1) Euler's 1^{st} law and (2) Euler's 2^{nd} law relative to the center of mass of the rod. The free body diagram of the rod is shown in Fig. 5-22.
Using Fig. 5-22, we have

$$\begin{aligned}\mathbf{P} &= P\mathbf{E}_x \\ \mathbf{N} &= N\mathbf{E}_y \\ \mathbf{R} &= R\mathbf{E}_x \\ m\mathbf{g} &= mg\mathbf{E}_y\end{aligned}$$

The resultant force acting on the particle is then given as

$$\mathbf{F} = \mathbf{P} + \mathbf{R} + \mathbf{N} + m\mathbf{g} = (P+R)\mathbf{E}_x + (N+mg)\mathbf{E}_y. \tag{5.563}$$

Applying Euler's first law to the sphere, we have

$$\mathbf{F} = m\,{}^{\mathcal{G}}\bar{\mathbf{a}} = m\,{}^{\mathcal{G}}\mathbf{a}_O \tag{5.564}$$

which implies that

$$(P+R)\mathbf{E}_x + (N+mg)\mathbf{E}_y = m\ddot{x}\mathbf{E}_x. \tag{5.565}$$

We then obtain the following two equations:

$$\begin{aligned}P + R &= m\ddot{x}. &(5.566)\\ N + mg &= 0. &(5.567)\end{aligned}$$

Next, applying Euler's second law relative to the center of mass of the sphere, we have

$$\bar{\mathbf{M}} = \frac{{}^{\mathcal{G}}d}{dt}\left({}^{\mathcal{G}}\bar{\mathbf{H}}\right) \Longleftrightarrow \mathbf{M}_O = \frac{{}^{\mathcal{G}}d}{dt}\left({}^{\mathcal{G}}\mathbf{H}_O\right) \tag{5.568}$$

Now we already have the rate of change of angular momentum relative to the center of mass of the rod from Eq. (5.562). The moment relative to the center of mass of the sphere is given as

$$\mathbf{M}_O = (\mathbf{r}_A - \mathbf{r}_O) \times \mathbf{P} + (\mathbf{r}_P - \mathbf{r}_O) \times \mathbf{R}, \tag{5.569}$$

where A is the point at which the force **P** acts and P is the point where the forces **R** and **N** act.

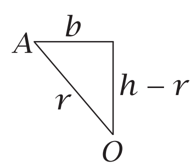

Figure 5-23 Location of Forces for Question 5–16.

Examining Fig. 5-23, it is seen that

$$\begin{aligned} \mathbf{r}_A - \mathbf{r}_O &= -b\mathbf{E}_x - (h-r)\mathbf{E}_y, & (5.570) \\ \mathbf{r}_P - \mathbf{r}_O &= r\mathbf{E}_y, & (5.571) \end{aligned}$$

where $b = \sqrt{r^2 - (h-r)^2}$. Therefore,

$$\begin{aligned} \mathbf{M}_O &= (-b\mathbf{E}_x - (h-r)\mathbf{E}_y) \times P\mathbf{E}_x + r\mathbf{E}_y \times R\mathbf{E}_x \\ &= (h-r)P\mathbf{E}_z - rR\mathbf{E}_z = [(h-r)P - rR]\,\mathbf{E}_z. \end{aligned} \tag{5.572}$$

Then, setting $\mathbf{M}_O$ equal to ${}^{\mathcal{G}}d({}^{\mathcal{G}}\mathbf{H}_O)/dt$, we obtain

$$(h-r)P - rR = \frac{2}{5}mr\ddot{x}. \tag{5.573}$$

Solving Eq. (5.566) for R and substituting the result into Eq. (5.573), we obtain

$$(h-r)P - r(m\ddot{x} - P) = \frac{2}{5}mr\ddot{x} \tag{5.574}$$

Simplifying this last result, we obtain the differential equation of motion as

$$\ddot{x} = \frac{5hP}{7mr}. \tag{5.575}$$

Question 5–17

A uniform disk of mass m and radius r rolls without slip along the inside of a fixed circular track of radius R as shown in Fig. P5-17. The angles θ and ϕ measure the position of the center of the disk and the angle of rotation of the disk, respectively, relative to the vertically downward direction. Knowing that the angles θ and ϕ are simultaneously zero and that gravity acts downward, determine the differential equation of motion for the disk *in terms of the angle* θ.

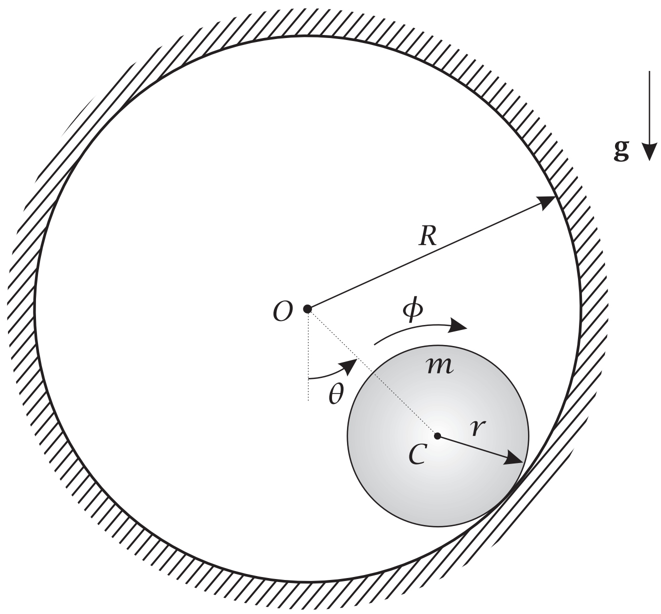

Figure P5-22

Solution to Question 5–17

Kinematics

First, let $\mathcal{F}$ be a fixed reference frame. Then, choose the following coordinate system fixed in $\mathcal{F}$:

	Origin at O	
$\mathbf{E}_x$	=	Along OC When $\theta = 0$
$\mathbf{E}_z$	=	Out of Page
$\mathbf{E}_y$	=	$\mathbf{E}_z \times \mathbf{E}_x$

Next, let $\mathcal{R}$ be a reference frame fixed to the direction of OC. Then, choose the following coordinate system fixed in reference frame $\mathcal{R}$:

$$\begin{array}{ccc} & \text{Origin at } O & \\ \mathbf{e}_r & = & \text{Along OC} \\ \mathbf{e}_z & = & \text{Out Page} \\ \mathbf{e}_\theta & = & \mathbf{e}_z \times \mathbf{e}_r \end{array}$$

Third, let $\mathcal{D}$ be the disk. Then, choose the following coordinate system fixed in reference frame $\mathcal{R}$:

$$\begin{array}{ccc} & \text{Origin at } O & \\ \mathbf{u}_r & = & \text{Along OC} \\ \mathbf{u}_z & = & \text{Out Page} \\ \mathbf{u}_\phi & = & \mathbf{u}_z \times \mathbf{u}_r \end{array}$$

The geometry of the bases $\{\mathbf{E}_x, \mathbf{E}_y, \mathbf{E}_z\}$ and $\{\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_z\}$ is shown in Fig. 5-24.

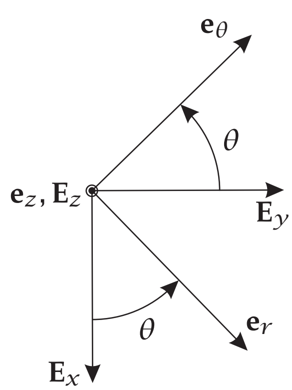

Figure 5-24 Unit Vertical Direction for Question 5–17.

Using Fig. 5-24, we have that

$$\mathbf{E}_x = \cos\theta\,\mathbf{e}_r - \sin\theta\,\mathbf{e}_\theta \tag{5.576}$$

$$\mathbf{E}_y = \sin\theta\,\mathbf{e}_r + \cos\theta\,\mathbf{e}_\theta \tag{5.577}$$

Next, the position of the center of mass of the disk is given as

$$\bar{\mathbf{r}} = \mathbf{r}_C = (R - r)\mathbf{e}_r \tag{5.578}$$

Furthermore, the angular velocity of reference frame $\mathcal{R}$ in reference frame $\mathcal{F}$ is given as

$$^{\mathcal{F}}\boldsymbol{\omega}^{\mathcal{R}} = \dot{\theta}\mathbf{E}_z \tag{5.579}$$

Then, applying the rate of change transport theorem between reference frame $\mathcal{R}$ and reference frame $\mathcal{F}$, we obtain the velocity of the center of mass of the disk as

$$^{\mathcal{F}}\bar{\mathbf{v}} = \frac{^{\mathcal{F}}d\bar{\mathbf{r}}}{dt} = \frac{^{\mathcal{R}}d\bar{\mathbf{r}}}{dt} + {}^{\mathcal{F}}\boldsymbol{\omega}^{\mathcal{R}} \times \bar{\mathbf{r}} \tag{5.580}$$

Now we have that

$$\begin{aligned} \frac{{}^{\mathcal{R}}d\bar{\mathbf{r}}}{dt} &= \mathbf{0} && (5.581) \\ {}^{\mathcal{F}}\boldsymbol{\omega}^{\mathcal{R}} \times \bar{\mathbf{r}} &= \dot{\theta}\mathbf{e}_z \times (R-r)\mathbf{e}_r = (R-r)\dot{\theta}\mathbf{e}_\theta && (5.582) \end{aligned}$$

Adding Eq. (5.581) and Eq. (5.582), we obtain the velocity of the center of mass of the rod in reference frame $\mathcal{F}$ as Differentiating $\mathbf{v}_C$ in Eq. (5.580), we obtain

$$ {}^{\mathcal{F}}\bar{\mathbf{v}} = (R-r)\dot{\theta}\mathbf{e}_\theta \quad (5.583)$$

We can obtain a second expression for ${}^{\mathcal{F}}\bar{\mathbf{v}}$ as follows. Since point P and point C are both fixed in the disk, we have that

$$ {}^{\mathcal{F}}\bar{\mathbf{v}} - {}^{\mathcal{F}}\mathbf{v}_P = {}^{\mathcal{F}}\boldsymbol{\omega}^{\mathcal{D}} \times (\bar{\mathbf{r}} - \mathbf{r}_P) \quad (5.584)$$

where ${}^{\mathcal{F}}\boldsymbol{\omega}^{\mathcal{D}}$ is the angular velocity of the disk in reference frame $\mathcal{F}$. Now, since ϕ describes the rotation of the disk relative to the vertical and ϕ is measured in the direction opposite the angle θ, we have that

$$ {}^{\mathcal{F}}\boldsymbol{\omega}^{\mathcal{D}} = -\dot{\phi}\mathbf{e}_z \quad (5.585)$$

Then, noting that $\bar{\mathbf{r}} - \mathbf{r}_P = -r\mathbf{e}_r$, we have that

$$ {}^{\mathcal{F}}\bar{\mathbf{v}} - {}^{\mathcal{F}}\mathbf{v}_P = -\dot{\phi}\mathbf{e}_z \times (-r\mathbf{e}_r) = r\dot{\phi}\mathbf{e}_\theta \quad (5.586)$$

Furthermore, since the disk rolls without slip, we have that

$$ {}^{\mathcal{F}}\mathbf{v}_P = \mathbf{0} \quad (5.587)$$

Therefore,

$$ {}^{\mathcal{F}}\bar{\mathbf{v}} = r\dot{\phi}\mathbf{e}_\theta \quad (5.588)$$

Then, setting the result of Eq. (5.588) equal to the result of Eq. (5.583), we obtain

$$ r\dot{\phi} = (R-r)\dot{\theta} \quad (5.589)$$

Solving Eq. (5.589) for $\dot{\phi}$, we obtain

$$ \dot{\phi} = \frac{R-r}{r}\dot{\theta} \quad (5.590)$$

Next, applying the rate of change transport theorem to ${}^{\mathcal{F}}\bar{\mathbf{v}}$ using the expression for ${}^{\mathcal{F}}\bar{\mathbf{v}}$ in Eq. (5.583, we obtain the acceleration of the center of mass of the disk in reference frame $\mathcal{F}$ as

$$ {}^{\mathcal{F}}\bar{\mathbf{a}} = \frac{{}^{\mathcal{F}}d}{dt}\left({}^{\mathcal{F}}\bar{\mathbf{a}}\right) = \frac{{}^{\mathcal{R}}d}{dt}\left({}^{\mathcal{F}}\bar{\mathbf{v}}\right) + {}^{\mathcal{F}}\boldsymbol{\omega}^{\mathcal{R}} \times {}^{\mathcal{F}}\bar{\mathbf{v}} \quad (5.591)$$

Now we have that

$$\begin{aligned} {}^{\mathcal{R}}\frac{d}{dt}\left({}^{\mathcal{F}}\bar{\mathbf{v}}\right) &= (R-r)\ddot{\theta}\mathbf{e}_\theta &\quad (5.592)\\ {}^{\mathcal{F}}\boldsymbol{\omega}^{\mathcal{R}} \times {}^{\mathcal{F}}\bar{\mathbf{v}} &= \dot{\theta}\mathbf{e}_z \times (a\dot{\theta}\mathbf{e}_\theta) = -(R-r)\dot{\theta}^2\mathbf{e}_r &\quad (5.593) \end{aligned}$$

Adding Eq. (5.592) and Eq. (5.593), we obtain the acceleration of the center of mass of the rod in reference frame $\mathcal{F}$ as

$$ {}^{\mathcal{F}}\bar{\mathbf{a}} = -(R-r)\dot{\theta}^2\mathbf{e}_r + (R-r)\ddot{\theta}\mathbf{e}_\theta \tag{5.594}$$

Finally, since point C and point P are both fixed to the disk, the acceleration of point P in reference frame $\mathcal{F}$ is obtained as

$$ {}^{\mathcal{F}}\mathbf{a}_P = {}^{\mathcal{F}}\bar{\mathbf{a}} + {}^{\mathcal{F}}\boldsymbol{\alpha}^{\mathcal{D}} \times (\mathbf{r}_P - \bar{\mathbf{r}}) + {}^{\mathcal{F}}\boldsymbol{\omega}^{\mathcal{D}}\left[{}^{\mathcal{F}}\boldsymbol{\omega}^{\mathcal{D}} \times (\mathbf{r}_P - \bar{\mathbf{r}})\right] \tag{5.595}$$

Now we have that

$$ {}^{\mathcal{F}}\boldsymbol{\alpha}^{\mathcal{D}} = \frac{{}^{\mathcal{F}}d}{dt}\left({}^{\mathcal{F}}\boldsymbol{\omega}^{\mathcal{D}}\right) = -\ddot{\phi}\mathbf{e}_z = -\frac{R-r}{r}\ddot{\theta}\mathbf{e}_z \tag{5.596}$$

Therefore

$$ {}^{\mathcal{F}}\mathbf{a}_P = {}^{\mathcal{F}}\bar{\mathbf{a}} - \frac{R-r}{r}\ddot{\theta}\mathbf{e}_z \times (r\mathbf{e}_r) - \frac{R-r}{r}\dot{\theta}\mathbf{e}_z \times \left[-\frac{R-r}{r}\dot{\theta}\mathbf{e}_z \times (r\mathbf{e}_r)\right] \tag{5.597}$$

Simplifying Eq. (5.597), we obtain

$$ {}^{\mathcal{F}}\mathbf{a}_P = {}^{\mathcal{F}}\bar{\mathbf{a}} - (R-r)\ddot{\theta}\mathbf{e}_\theta - \frac{(R-r)^2\dot{\theta}^2}{r}\mathbf{e}_r \tag{5.598}$$

Then, substituting the expression for ${}^{\mathcal{F}}\bar{\mathbf{a}}$ from Eq. (5.594) into Eq. (5.598), we obtain

$$ {}^{\mathcal{F}}\mathbf{a}_P = -(R-r)\dot{\theta}^2\mathbf{e}_r - \frac{(R-r)^2\dot{\theta}^2}{r}\mathbf{e}_r \tag{5.599}$$

Simplifying Eq. (5.599), we obtain ${}^{\mathcal{F}}\mathbf{a}_P$ as

$$ {}^{\mathcal{F}}\mathbf{a}_P = -(R-r)\left[1 + \frac{R-r}{r}\right]\dot{\theta}^2\mathbf{e}_r \tag{5.600}$$

Kinetics

The free body diagram of the disk is shown in Fig. 5-25.
Using Fig. 5-25, it is seen that the following forces act on the disk

$$\begin{aligned} \mathbf{N} &= \text{Reaction Force of Track on Disk} \\ \mathbf{F}_R &= \text{Rolling Force of Track on Disk} \\ m\mathbf{g} &= \text{Force of Gravity} \end{aligned}$$

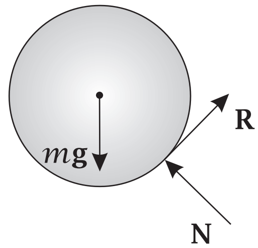

Figure 5-25 Free Body Diagram for Question 5–17.

Now from the geometry we have that

$$\mathbf{N} = N\mathbf{e}_r \quad (5.601)$$
$$\mathbf{F}_R = F_R\mathbf{e}_\theta \quad (5.602)$$
$$m\mathbf{g} = mg\mathbf{E}_x \quad (5.603)$$

Using the expression for $\mathbf{E}_x$ from Eq. (5.576), the force of gravity can be written in terms of the basis $\{\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_z\}$ as

$$m\mathbf{g} = mg\cos\theta\,\mathbf{e}_r - mg\sin\theta\,\mathbf{e}_\theta \quad (5.604)$$

Now we will use the following three methods to determine the differential equation of motion: (1) Euler's $1^{st} law$ and Euler's 2^{nd} law relative to the center of mass of the disk, (2) Euler's 2^{nd} law relative to the instantaneous point of contact of the disk with the track and (3) the alternate form of the work-energy theorem for a rigid body.

Method 1: Euler's 1^{st} Law and Euler's 2^{nd} Law Relative to Center of Mass of Disk

Applying Euler's 1^{st} law, we have that

$$\mathbf{F} = m\,{}^{\mathcal{F}}\bar{\mathbf{a}} \quad (5.605)$$

Now, using the forces as given in Eq. (5.601), Eq. (5.602), and Eq. (5.604), the resultant force acting on the disk is given as

$$\mathbf{F} = \mathbf{N} + \mathbf{F}_R + m\mathbf{g} = N\mathbf{e}_r + F_R\mathbf{e}_\theta + mg\cos\theta\,\mathbf{e}_r - mg\sin\theta\,\mathbf{e}_\theta \quad (5.606)$$

Simplifying Eq. (5.606), we obtain

$$\mathbf{F} = (N + mg\cos\theta)\mathbf{e}_r + (F_R - mg\sin\theta)\mathbf{e}_\theta \quad (5.607)$$

Then, setting $\mathbf{F}$ in Eq. (5.607) equal to $m\,{}^{\mathcal{F}}\bar{\mathbf{a}}$ using the expression for ${}^{\mathcal{F}}\bar{\mathbf{a}}$ from Eq. (5.594), we obtain

$$(N + mg\cos\theta)\mathbf{e}_r + (F_R - mg\sin\theta)\mathbf{e}_\theta = -m(R-r)\dot{\theta}^2\mathbf{e}_r + m(R-r)\ddot{\theta}\mathbf{e}_\theta \quad (5.608)$$

Equating components in Eq. (5.608), we obtain

$$\begin{aligned} N + mg\cos\theta &= -m(R-r)\dot{\theta}^2 && (5.609)\\ F_R - mg\sin\theta &= m(R-r)\ddot{\theta} && (5.610) \end{aligned}$$

Applying Euler's 2^{nd} law relative to the center of mass of the disk, we have that

$$\bar{\mathbf{M}} = \frac{^{\mathcal{F}}d}{dt}\left(^{\mathcal{F}}\bar{\mathbf{H}}\right) \quad (5.611)$$

Now we have that

$$^{\mathcal{F}}\bar{\mathbf{H}} = \bar{\mathbf{I}}^{\mathcal{D}} \cdot {^{\mathcal{F}}\boldsymbol{\omega}^{\mathcal{D}}} \quad (5.612)$$

Now since $\{\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_z\}$ is a principal-axis basis, we have that

$$\bar{\mathbf{I}}^{\mathcal{D}} = \bar{I}_{rr}\mathbf{u}_r \otimes \mathbf{u}_r + \bar{I}_{\phi\phi}\mathbf{u}_\phi \otimes \mathbf{u}_\phi + \bar{I}_{zz}\mathbf{u}_z \otimes \mathbf{u}_z \quad (5.613)$$

Then, using the expression for $^{\mathcal{F}}\boldsymbol{\omega}^{\mathcal{D}}$ from Eq. (5.585) and the expression for $\dot{\phi}$ from Eq. (5.590), we obtain

$$^{\mathcal{F}}\bar{\mathbf{H}} = -\bar{I}_{zz}\frac{R-r}{r}\dot{\theta}\mathbf{u}_z \quad (5.614)$$

Now we have for a uniform circular disk that

$$\bar{I}_{zz} = \frac{mr^2}{2} \quad (5.615)$$

Consequently, $^{\mathcal{F}}\bar{\mathbf{H}}$ becomes

$$^{\mathcal{F}}\bar{\mathbf{H}} = -\frac{mr^2}{2}\frac{R-r}{r}\dot{\theta}\mathbf{u}_z = -\frac{mr(R-r)}{2}\dot{\theta}\mathbf{u}_z \quad (5.616)$$

Computing the rate of change of $^{\mathcal{F}}\bar{\mathbf{H}}$ in reference frame $\mathcal{F}$, we obtain

$$\frac{^{\mathcal{F}}d}{dt}\left(^{\mathcal{F}}\bar{\mathbf{H}}\right) = -\frac{mr(R-r)}{2}\ddot{\theta}\mathbf{u}_z \quad (5.617)$$

Next, because the forces $\mathbf{N}$ and $m\mathbf{g}$ pass through the center of mass of the disk, the moment applied to the disk relative to the center of mass of the disk is due entirely to bfF_R and is given as

$$\bar{\mathbf{M}} = (\mathbf{r}_P - \bar{\mathbf{r}}) \times \mathbf{F}_R = r\mathbf{e}_r \times F_R\mathbf{e}_\theta = rF_R\mathbf{e}_z \quad (5.618)$$

Then, setting $\bar{\mathbf{M}}$ from Eq. (5.618) equal to $^{\mathcal{F}}d(^{\mathcal{F}}\bar{\mathbf{H}})/dt$, we obtain

$$-\frac{mr(R-r)}{2}\ddot{\theta} = rF_R \quad (5.619)$$

Simplifying Eq. (5.619), we obtain

$$F_R = -\frac{m(R-r)}{2}\ddot{\theta} \tag{5.620}$$

The differential equation of motion can now be obtained using Eq. (5.610) and Eq. (5.620). Substituting the expression for F_R from Eq. (5.620) into Eq. (5.610), we obtain

$$-\frac{m(R-r)}{2}\ddot{\theta} - mg\sin\theta = m(R-r)\ddot{\theta} \tag{5.621}$$

Simplifying Eq. (5.621), we obtain the differential equation of motion as

$$\frac{3}{2}m(R-r)\ddot{\theta} + mg\sin\theta = 0 \tag{5.622}$$

Method 2: Euler's 2^{nd} Law Relative to Instantaneous Point of Contact

Since the point of contact is an arbitrary point, we have that

$$\mathbf{M}_P - (\bar{\mathbf{r}} - \mathbf{r}_P) \times m\,{}^{\mathcal{F}}\mathbf{a}_P = \frac{{}^{\mathcal{F}}d}{dt}\left({}^{\mathcal{F}}\mathbf{H}_P\right) \tag{5.623}$$

First, the angular momentum relative to point P is given as

$${}^{\mathcal{F}}\mathbf{H}_P = {}^{\mathcal{F}}\bar{\mathbf{H}} + (\bar{\mathbf{r}} - \mathbf{r}_P) \times m({}^{\mathcal{F}}\bar{\mathbf{v}} - {}^{\mathcal{F}}\mathbf{v}_P) \tag{5.624}$$

Then, using the fact that $\bar{\mathbf{r}} - \mathbf{r}_P = -r\mathbf{e}_r$, the fact that ${}^{\mathcal{F}}\bar{\mathbf{v}} - {}^{\mathcal{F}}\mathbf{v}_P = {}^{\mathcal{F}}\bar{\mathbf{v}} = (R-r)\dot{\theta}\mathbf{e}_\theta$, and the expression for ${}^{\mathcal{F}}\bar{\mathbf{H}}$ from Eq. (5.616), we obtain ${}^{\mathcal{F}}\mathbf{H}_P$ as

$${}^{\mathcal{F}}\mathbf{H}_P = -\frac{mr(R-r)}{2}\dot{\theta}\mathbf{u}_z + (-r\mathbf{e}_r) \times m(R-r)\dot{\theta}\mathbf{e}_\theta = -\frac{3mr(R-r)}{2}\dot{\theta}\mathbf{u}_z \tag{5.625}$$

Then, computing the rate of change of ${}^{\mathcal{F}}\mathbf{H}_P$ in reference frame $\mathcal{F}$, we obtain

$$\frac{{}^{\mathcal{F}}d}{dt}\left({}^{\mathcal{F}}\mathbf{H}_P\right) = -\frac{3mr(R-r)}{2}\ddot{\theta}\mathbf{u}_z \tag{5.626}$$

Next, since the forces $\mathbf{N}$ and $\mathbf{F}_R$ pass through point P, the moment acting on the disk relative to point P is due entirely to gravity and is given as

$$\mathbf{M}_P = (\bar{\mathbf{r}} - \mathbf{r}_P) \times m\mathbf{g} = -r\mathbf{e}_r \times (mg\cos\theta\,\mathbf{e}_r - mg\sin\theta\,\mathbf{e}_\theta) = mgr\sin\theta\,\mathbf{e}_z \tag{5.627}$$

Finally, the inertial moment is given as

$$-(\bar{\mathbf{r}} - \mathbf{r}_P) \times m\,{}^{\mathcal{F}}\mathbf{a}_P = -(-r\mathbf{e}_r) \times m\left[-(R-r)\left[1 + \frac{R-r}{r}\right]\dot{\theta}^2\mathbf{e}_r\right] = \mathbf{0} \tag{5.628}$$

Substituting the results of Eq. (5.626), Eq. (5.627), and Eq. (5.628) into Eq. (5.623), we obtain

$$mgr\sin\theta = -\frac{3mr(R-r)}{2}\ddot{\theta} \tag{5.629}$$

Simplifying Eq. (5.629), we obtain the differential equation of motion as

$$\frac{3}{2}m(R-r)\ddot{\theta} + mg\sin\theta = 0 \tag{5.630}$$

Method 3: Alternate Form of Work-Energy Theorem for a Rigid Body

Of the three forces acting on the disk, we know that gravity is conservative. Furthermore, since the velocity of point P in reference frame $\mathcal{F}$ is zero, we know that neither $\mathbf{N}$ nor bfF_R does work. Finally, since no pure torques act on the disk, we know that the work done by all non-conservative forces and non-conservative pure torques is zero. Therefore, we have that

$$\frac{d}{dt}\left({}^{\mathcal{F}}E\right) = 0 \tag{5.631}$$

Now the total energy in reference frame $\mathcal{F}$ is given as

$${}^{\mathcal{F}}E = {}^{\mathcal{F}}T + {}^{\mathcal{F}}U \tag{5.632}$$

First, the kinetic energy of the disk in reference frame $\mathcal{F}$ is given as

$${}^{\mathcal{F}}T = \frac{1}{2}m{}^{\mathcal{F}}\bar{\mathbf{v}} \cdot {}^{\mathcal{F}}\bar{\mathbf{v}} + \frac{1}{2}{}^{\mathcal{F}}\bar{\mathbf{H}} \cdot {}^{\mathcal{F}}\boldsymbol{\omega}^{\mathcal{D}} \tag{5.633}$$

Using the expression for ${}^{\mathcal{F}}\bar{\mathbf{v}}$ from Eq. (5.583), we have that

$$\frac{1}{2}m{}^{\mathcal{F}}\bar{\mathbf{v}} \cdot {}^{\mathcal{F}}\bar{\mathbf{v}} = \frac{1}{2}m(R-r)^2\dot{\theta}^2 \tag{5.634}$$

Next, using the expression for ${}^{\mathcal{F}}\bar{\mathbf{H}}$, the expression for ${}^{\mathcal{F}}\boldsymbol{\omega}^{\mathcal{D}}$ from Eq. (5.585), and the expression for $\dot{\phi}$ from Eq. (5.590), we have that

$$\frac{1}{2}{}^{\mathcal{F}}\bar{\mathbf{H}} \cdot {}^{\mathcal{F}}\boldsymbol{\omega}^{\mathcal{D}} = \frac{1}{2}\left[-\frac{mr^2}{2}\frac{R-r}{r}\dot{\theta}\mathbf{e}_z\right] \cdot \left[-\frac{R-r}{r}\dot{\theta}\mathbf{e}_z\right] = \frac{1}{4}m(R-r)^2\dot{\theta}^2 \tag{5.635}$$

Adding Eq. (5.634) and Eq. (5.635), we obtain the kinetic energy of the disk in reference frame $\mathcal{F}$ as

$${}^{\mathcal{F}}T = \frac{3}{4}m(R-r)^2\dot{\theta}^2 \tag{5.636}$$

Next, the potential energy of the disk in reference frame $\mathcal{F}$ is given as

$$\begin{aligned} {}^{\mathcal{F}}U = {}^{\mathcal{F}}U_g &= -m\mathbf{g} \cdot \bar{\mathbf{r}} \\ &= -(mg\cos\theta\,\mathbf{e}_r - mg\sin\theta\,\mathbf{e}_\theta) \cdot (R-r)\mathbf{e}_r = -mg(R-r)\cos\theta \end{aligned} \tag{5.637}$$

The total energy in reference frame $\mathcal{F}$ is then given as

$${}^{\mathcal{F}}E = {}^{\mathcal{F}}T + {}^{\mathcal{F}}U = \frac{3}{4}m(R-r)^2\dot{\theta}^2 - mg(R-r)\cos\theta \tag{5.638}$$

Computing the rate of change of ${}^{\mathcal{F}}E$ and setting the result equal to zero, we obtain

$$\frac{d}{dt}\left({}^{\mathcal{F}}E\right) = \frac{3}{2}m(R-r)^2\dot{\theta}\ddot{\theta} + mg(R-r)\dot{\theta}\sin\theta = 0 \tag{5.639}$$

Eq. (5.639) can be re-written as

$$(R-r)\dot{\theta}\left[\frac{3}{2}m(R-r)\ddot{\theta}+mg\sin\theta\right]=0 \tag{5.640}$$

Observing that $\dot{\theta}$ is not zero as a function of time, we obtain the differential equation of motion as

$$\frac{3}{2}m(R-r)\ddot{\theta}+mg\sin\theta\ =0 \tag{5.641}$$

addtocounterpfigure4

Question 5–20

A uniform circular disk of mass m and radius r rolls without slip along a plane inclined at a constant angle β with horizontal as shown in Fig. P5-20. Attached to the disk at the point A (where A lies in the direction of PO) is a linear spring with spring constant K and a nonlinear damper with damping constant c. The damper exerts a force of the form

$$\mathbf{F}_d = -c\|\mathbf{v}_A\|^3 \frac{\mathbf{v}_A}{\|\mathbf{v}_A\|}$$

where $\mathbf{v}_A$ is the velocity of point A. Knowing that the spring is unstretched when the angle θ is zero and that gravity acts downward, determine the differential equation of motion for the disk.

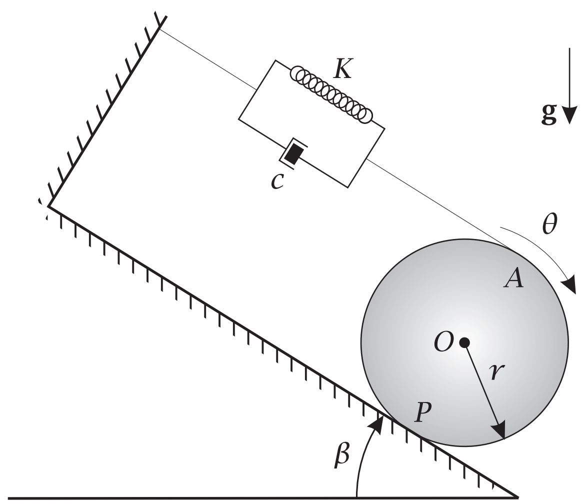

Figure P5-23

Solution to Question 5–20

Kinematics

First, let $\mathcal{F}$ be a fixed reference frame. Then, choose the following coordinate system fixed in reference frame $\mathcal{F}$:

	Origin at Point O When $t = 0$	
$\mathbf{E}_x$	=	Along Incline
$\mathbf{E}_z$	=	Into Page
$\mathbf{E}_y$	=	$\mathbf{E}_z \times \mathbf{E}_x$

Next, let $\mathcal{D}$ be the disk. Then, choose the following coordinate system fixed in reference frame $\mathcal{F}$:

$$
\begin{array}{ccc}
 & \text{Origin at Point } O & \\
 & \text{When } t = 0 & \\
\mathbf{e}_r & = & \text{Fixed to } \mathcal{D} \\
\mathbf{e}_z & = & \mathbf{E}_z \\
\mathbf{e}_\theta & = & \mathbf{e}_z \times \mathbf{e}_r
\end{array}
$$

Now, because the disk rolls without slip along a fixed surface, we have that

$$ {}^{\mathcal{F}}\mathbf{v}_P = \mathbf{0} \tag{5.642} $$

Then, from kinematics of rigid bodies, the velocity of point O is given as

$$ {}^{\mathcal{F}}\mathbf{v}_O = {}^{\mathcal{F}}\mathbf{v}_P + {}^{\mathcal{F}}\boldsymbol{\omega}^{\mathcal{R}} \times (\mathbf{r}_O - \mathbf{r}_P) \tag{5.643} $$

where $\mathcal{R}$ is the reference frame of the disk. Now, since the angle θ describes the amount that the disk has rotated since time $t = 0$, we have that

$$ {}^{\mathcal{F}}\boldsymbol{\omega}^{\mathcal{R}} = \dot{\theta}\mathbf{e}_z \tag{5.644} $$

Furthermore, noting that $\mathbf{r}_O - \mathbf{r}_P = -r\mathbf{E}_y$, we obtain $\mathbf{v}_O$ as

$$ {}^{\mathcal{F}}\mathbf{v}_O = \dot{\theta}\mathbf{E}_z \times (-r\mathbf{E}_y) = r\dot{\theta}\mathbf{E}_x \tag{5.645} $$

Differentiating ${}^{\mathcal{F}}\mathbf{v}_O$ in reference frame $\mathcal{F}$, we obtain the acceleration of point O in reference frame $\mathcal{F}$ as

$$ {}^{\mathcal{F}}\mathbf{a}_O = \frac{{}^{\mathcal{F}}d}{dt}\left({}^{\mathcal{F}}\mathbf{v}_O\right) = r\ddot{\theta}\mathbf{E}_x \tag{5.646} $$

Finally, we obtain the velocity of point A in reference frame $\mathcal{F}$ as

$$ {}^{\mathcal{F}}\mathbf{v}_A = {}^{\mathcal{F}}\mathbf{v}_P + {}^{\mathcal{F}}\boldsymbol{\omega}^{\mathcal{R}} \times (\mathbf{r}_A - \mathbf{r}_P) \tag{5.647} $$

Noting that $\mathbf{r}_A - \mathbf{r}_P = -2r\mathbf{E}_y$, we obtain ${}^{\mathcal{F}}\mathbf{v}_A$ as

$$ {}^{\mathcal{F}}\mathbf{v}_A = \dot{\theta}\mathbf{E}_z \times (-2r\mathbf{E}_y) = 2r\dot{\theta}\mathbf{E}_x \tag{5.648} $$

Kinetics

The free body diagram of the disk is shown in Fig. 5-26. It can be seen that the forces acting on the disk are as follows:

$$
\begin{array}{lcl}
m\mathbf{g} & = & \text{Force of Gravity} \\
\mathbf{R} & = & \text{Rolling Force} \\
\mathbf{N} & = & \text{Normal Force Applied by Incline on Disk} \\
\mathbf{F}_s & = & \text{Spring Force} \\
\mathbf{F}_d & = & \text{Damping Force}
\end{array}
$$

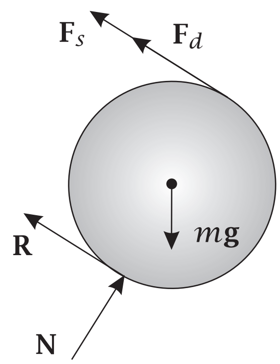

Figure 5-26 Free Body Diagram for Question 6.11

Now we have the following

$$\begin{aligned} \mathbf{R} &= R\mathbf{E}_x \\ \mathbf{N} &= N\mathbf{E}_y \\ m\mathbf{g} &= mg\mathbf{u}_v \end{aligned} \tag{5.649}$$

where $\mathbf{u}_v$ is the unit vector in the vertically downward direction. Using the geometry shown in Fig. 5-27, we have that

$$\mathbf{u}_v = \sin\theta\,\mathbf{E}_x + \cos\theta\,\mathbf{E}_y \tag{5.650}$$

The force of gravity is then given as

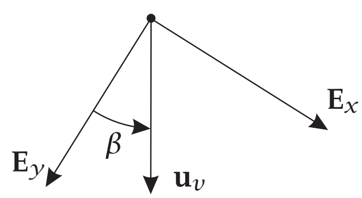

Figure 5-27 Downward Direction $\mathbf{u}_V$ in Terms of Basis $\{\mathbf{E}_x, \mathbf{E}_y\}$ for Question 6.11

$$m\mathbf{g} = mg\sin\beta\mathbf{E}_x + mg\cos\beta\mathbf{E}_y \tag{5.651}$$

Next, we need the spring force. We know that the spring force has the form

$$\mathbf{F}_s = -K\left[\|\boldsymbol{\rho}\| - L\right]\frac{\boldsymbol{\rho}}{\|\boldsymbol{\rho}\|} \tag{5.652}$$

where $\boldsymbol{\rho}$ is the position of the spring measured from its unstretched length L. Now for this problem we have that

$$\boldsymbol{\rho} = \mathbf{r}_A - \mathbf{r}_A(t = 0) \tag{5.653}$$

We can get $\mathbf{r}_A$ using $\mathbf{v}_A$ from Eq. (5.648). Noting that $\mathbf{r}_A(t=0) = -r\mathbf{E}_y$, we have that

$$\mathbf{r}_A = 2r\theta\mathbf{E}_x - r\mathbf{E}_y \tag{5.654}$$

Therefore, $\boldsymbol{\rho}$ is given as

$$\boldsymbol{\rho} = 2r\theta\mathbf{E}_x - r\mathbf{E}_y - (-r\mathbf{E}_y) = 2r\theta\mathbf{E}_x \tag{5.655}$$

Furthermore, the spring is unstretched when $\theta = 0$ which implies that $L = 0$. Therefore, we obtain the spring force as

$$\mathbf{F}_s = -2Kr\theta\mathbf{E}_x \tag{5.656}$$

Finally, the damping force is given as

$$\mathbf{F}_d = -c\|{}^{\mathcal{F}}\mathbf{v}_A\|^3 \frac{{}^{\mathcal{F}}\mathbf{v}_A}{\|{}^{\mathcal{F}}\mathbf{v}_A\|} \tag{5.657}$$

Using ${}^{\mathcal{F}}\mathbf{v}_A$ from Eq. (5.648), we obtain $\mathbf{F}_d$ as

$$\mathbf{F}_d = -c(2r\dot{\theta})^3\mathbf{E}_x = -8cr^3\dot{\theta}^3\mathbf{E}_x \tag{5.658}$$

Now that we have all of the forces, this problem can be solved by performing a moment balance about the point of contact, P. Since P is not a fixed point, we have that

$$\mathbf{M}_P - (\bar{\mathbf{r}} - \mathbf{r}_P) \times m\,{}^{\mathcal{F}}\mathbf{a}_P = \frac{{}^{\mathcal{F}}d}{dt}\left({}^{\mathcal{F}}\mathbf{H}_P\right) \tag{5.659}$$

Noting that $\bar{\mathbf{r}} = \mathbf{r}_O$, Eq. (5.659) can be written as

$$\mathbf{M}_P - (\mathbf{r}_O - \mathbf{r}_P) \times m\,{}^{\mathcal{F}}\mathbf{a}_P = \dot{\mathbf{H}}_P \tag{5.660}$$

Now we have that

$$\mathbf{r}_O - \mathbf{r}_P = -r\mathbf{E}_y \tag{5.661}$$

Furthermore, we have $\mathbf{a}_P$ as

$${}^{\mathcal{F}}\mathbf{a}_P = {}^{\mathcal{F}}\mathbf{a}_O + \boldsymbol{\alpha} \times (\mathbf{r}_P - \mathbf{r}_O) + {}^{\mathcal{F}}\boldsymbol{\omega}^{\mathcal{R}} \times \left[{}^{\mathcal{F}}\boldsymbol{\omega}^{\mathcal{R}} \times (\mathbf{r}_P - \mathbf{r}_O)\right] \tag{5.662}$$

Differentiating $\boldsymbol{\omega}$ in Eq. 5.644), we obtain the angular acceleration of the disk, $\boldsymbol{\alpha}$, as

$${}^{\mathcal{F}}\boldsymbol{\alpha}^{\mathcal{R}} = \frac{{}^{\mathcal{F}}d}{dt}\left({}^{\mathcal{F}}\boldsymbol{\omega}^{\mathcal{R}}\right) = \ddot{\theta}\mathbf{E}_z \tag{5.663}$$

Furthermore, using ${}^{\mathcal{F}}\mathbf{a}_O$ from Eq. (5.646), we obtain ${}^{\mathcal{F}}\mathbf{a}_P$ as

$${}^{\mathcal{F}}\mathbf{a}_P = r\ddot{\theta}\mathbf{E}_x + \ddot{\theta}\mathbf{E}_z \times r\mathbf{E}_y + \dot{\theta}\mathbf{E}_z \times \left[\dot{\theta}\mathbf{E}_z \times r\mathbf{E}_y\right] \tag{5.664}$$

Simplifying Eq. (5.664) gives

$${}^{\mathcal{F}}\mathbf{a}_P = r\ddot{\theta}\mathbf{E}_x - r\ddot{\theta}\mathbf{E}_x - r\dot{\theta}^2\mathbf{E}_y = -r\dot{\theta}^2\mathbf{E}_y \tag{5.665}$$

We then obtain

$$-(\mathbf{r}_O - \mathbf{r}_P) \times m^{\mathcal{F}}\mathbf{a}_P = r\mathbf{E}_y \times m(-r\dot{\theta}^2\mathbf{E}_y) = \mathbf{0} \tag{5.666}$$

Therefore, Eq. (5.659) reduces to

$$\mathbf{M}_P = \frac{^{\mathcal{F}}d}{dt}\left(^{\mathcal{F}}\mathbf{H}_P\right) \tag{5.667}$$

Examining the free body diagram in Fig. 5-26), we see that the forces **R** and **N** pass through point P. Consequently, the moment relative to point P is given as

$$\mathbf{M}_P = (\mathbf{r}_A - \mathbf{r}_P) \times (\mathbf{F}_s + \mathbf{F}_d) + (\mathbf{r}_O - \mathbf{r}_P) \times m\mathbf{g} \tag{5.668}$$

Now we note that $\mathbf{r}_A - \mathbf{r}_P = -2r\mathbf{E}_y$. Then, substituting the expressions for $\mathbf{F}_s$ from Eq. (5.656), $\mathbf{F}_d$ from Eq. (5.658), and $m\mathbf{g}$ from Eq. (5.651), we obtain $\mathbf{M}_P$ as

$$\mathbf{M}_P = (-2r\mathbf{E}_y)\times(-2Kr\theta\mathbf{E}_x - 8cr^3\dot{\theta}^3\mathbf{E}_x) + (-r\mathbf{E}_y)\times(mg\sin\beta\mathbf{E}_x + mg\cos\beta\mathbf{E}_y) \tag{5.669}$$

Eq. (5.669) simplifies to

$$\mathbf{M}_P = (-4Kr\theta - 16cr^4\dot{\theta}^3 + mgr\sin\beta)\mathbf{E}_z \tag{5.670}$$

Next, the angular momentum relative to point P is given as

$$^{\mathcal{F}}\mathbf{H}_P = {}^{\mathcal{F}}\bar{\mathbf{H}} + (\mathbf{r}_P - \bar{\mathbf{r}}) \times m(^{\mathcal{F}}\mathbf{v}_P - {}^{\mathcal{F}}\mathbf{v}bar) \tag{5.671}$$

Now since $\{\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_z\}$ is a principal-axis basis, we have

$$^{\mathcal{F}}\bar{\mathbf{H}} = \bar{\mathbf{I}}^{\mathcal{R}} = \bar{I}_{rr}\mathbf{e}_r \otimes \mathbf{e}_r + \bar{I}_{\theta\theta}\mathbf{e}_\theta \otimes \mathbf{e}_\theta + \bar{I}_{zz}\mathbf{e}_z \otimes \mathbf{e}_z \tag{5.672}$$

Consequently, we obtain $^{\mathcal{F}}\bar{\mathbf{H}}$ as

$$^{\mathcal{F}}\bar{\mathbf{H}} = \bar{I}_{zz}\dot{\theta}\mathbf{e}_z \tag{5.673}$$

We then obtain $^{\mathcal{F}}\mathbf{H}_P$ as

$$^{\mathcal{F}}\mathbf{H}_P = \bar{I}_{zz}\dot{\theta}\mathbf{e}_z + r\mathbf{E}_y \times m(-r\dot{\theta}\mathbf{e}_z) = \bar{I}_{zz} + mr^2\dot{\theta}\mathbf{e}_z \tag{5.674}$$

Noting that $\bar{I}_{zz} = mr^2$, we have

$$^{\mathcal{F}}\mathbf{H}_P = \frac{3}{2}mr^2\dot{\theta}\mathbf{e}_z \tag{5.675}$$

Differentiating $^{\mathcal{F}}\mathbf{H}_P$ in reference frame $\mathcal{F}$, we obtain

$$\frac{^{\mathcal{F}}d}{dt}\left(^{\mathcal{F}}\mathbf{H}_P\right) = \frac{3}{2}mr^2\ddot{\theta}\mathbf{e}_z \tag{5.676}$$

Then, setting $^{\mathcal{F}}d\left(^{\mathcal{F}}\mathbf{H}_P\right)/dt$ in Eq. (5.676) equal to $\mathbf{M}_P$ from Eq. (5.670), we obtain

$$\frac{3mr^2}{2}\ddot{\theta} = (-4Kr\theta - 16cr^4\dot{\theta}^3 + mgr\sin\beta) \tag{5.677}$$

from which we obtain the differential equation of motion as

$$\frac{3}{2}mr^2\ddot{\theta} + 16cr^4\dot{\theta}^3 + 4Kr\theta - mgr\sin\beta = 0 \tag{5.678}$$